PH444 (Electromagnetic Theory 1) Lectures on

MONDAY SLOT 2A (09:30-10:25)
TUESDAY SLOT 2B (10:35-11:30)
THURSDAY SLOT 2C (11:35-12:30)

Instructor : Kantimay Das Gupta : kdasgupta@phy

Reference texts:

D J Griffiths
Feynman Lectures: vol 2
Panofsky and Philips
Reitz, Milford and Christy
J D Jackson
A Zangwill

EVALUATION (typical) Quiz1=15 : Midsem=30 : Quiz2=15 : Endsem=40
1 Formula sheet + calculator allowed in all exams....no need to ask!

Course plan (~ 30 lectures + 10 tutorials)

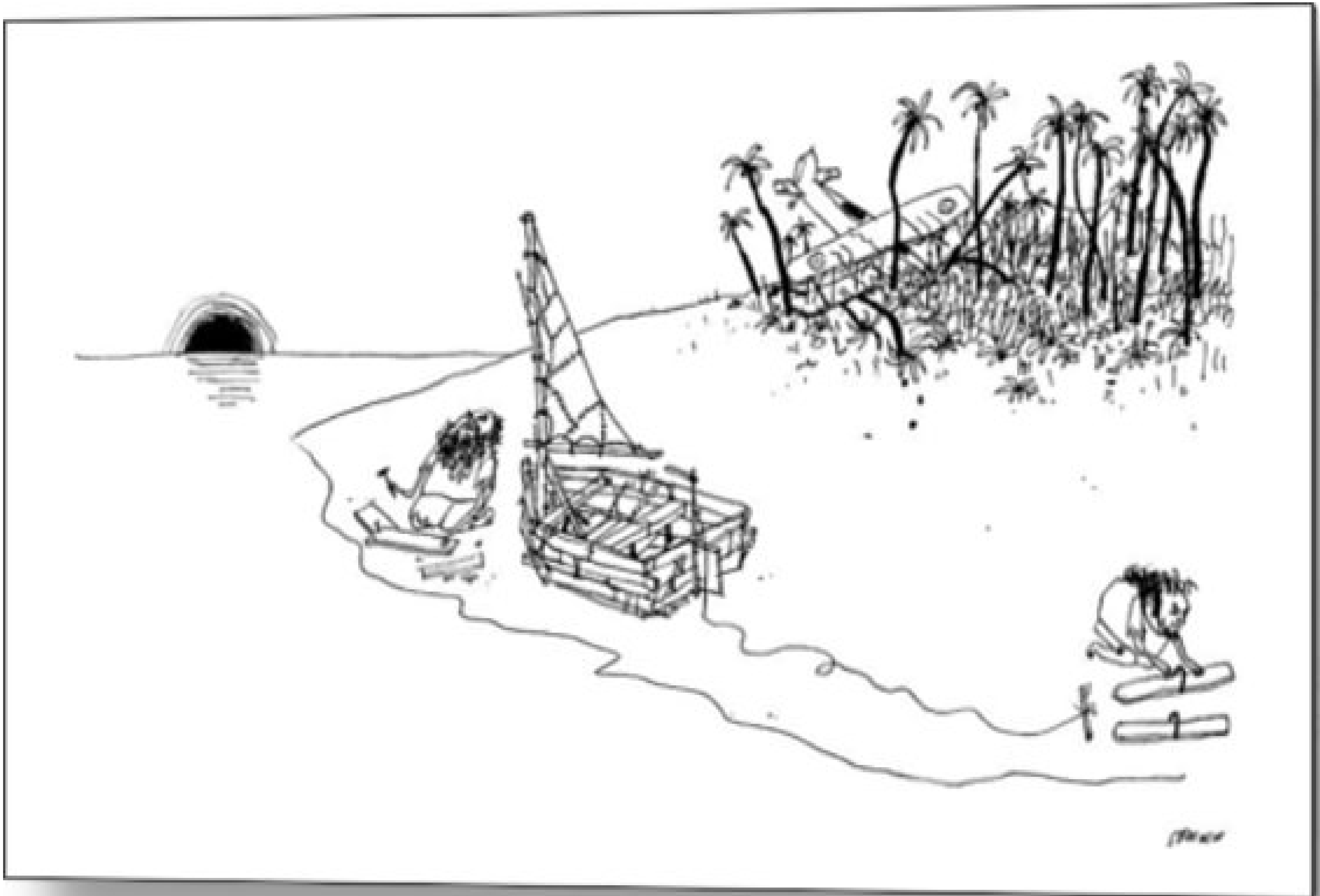
Bit of revision of co-ordinate systems
Electrostatics: Poisson Formula (2D), complex numbers & conformal mapping problems, how to go off-axis...
3D solutions in cylindrical systems (Bessel functions etc)
Green's theorem, solution for certain geometries

Multipoles. Dielectrics & Magnetic materials: Microscopic mechanisms, expressions for energy, defintions of E,D,B,H (what are the ambiguities?)

Energy, momentum & forces in EM, Stress Tensor and its uses.

Potential/fields of moving point charges (Leinard-Wiechart)
Radiation from accelerating charges, dipoles etc.
Antennas, transmission lines and waveguides.
Brehmsstralung, Synchrotron, Cerenkov radiation, free electron laser

A little bit of "desert island physics" : Why ?



In other words....."what if google is down“ ?
Working out from "first principles" makes things clearer!

Revision of grad, div, curl

How to derive the expressions in orthogonal curvilinear co-ordinates?

Curvilinear co-ordinates : quick revision

Writing the basic information about orthogonal co-ordinates....

$$d\vec{r} = \hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 du_1 + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 du_2 + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 du_3$$

$$ds^2 = ?$$

$$dV = ?$$

A shorthand compact way of writing co-ordinates

$$d\vec{r} = \sum \hat{\boldsymbol{\epsilon}}_{\boldsymbol{i}} h_i d\, u_i$$

Summation convention :

REPEATED INDEX IMPLIES SUMMATION

$$d\vec{r} = \hat{\boldsymbol{\epsilon}}_{\boldsymbol{i}} h_i d\, u_i$$

Exercise : See the list of co-ordinate systems given in Spiegel's vector analysis book....work out all the scale factors etc.

Curvilinear co-ordinates : Gradient

Given a scalar function $f\left(u_{1,}u_{2,}u_3\right)$ we want a vector such that

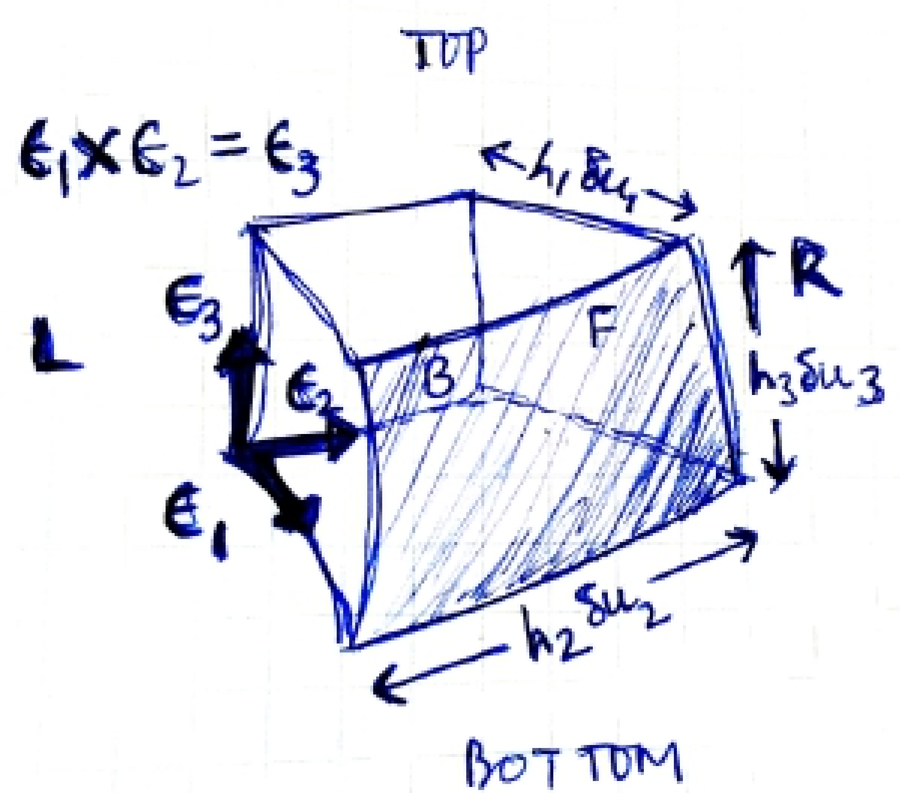
$$\begin{aligned}\delta f &= \vec{X}\,.\,\delta\vec{r}\\ &= \vec{X}\,.\left[\hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 \delta u_1 + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 \delta u_2 + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 \delta u_3\right]\\ &= \frac{\partial f}{\partial u_1}\delta u_1 + \frac{\partial f}{\partial u_2}\delta u_2 + \frac{\partial f}{\partial u_3}\delta u_3\end{aligned}$$

$$\vec{X} \equiv \nabla f = \left[\hat{\boldsymbol{\epsilon}}_{\mathbf{1}}\frac{1}{h_1}\frac{\partial f}{\partial u_1} + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}}\frac{1}{h_2}\frac{\partial f}{\partial u_2} + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}}\frac{1}{h_3}\frac{\partial f}{\partial u_3}\right]$$

For a given $|\delta\vec{r}|$ maximum change δf will happen
If the step is taken along the direction of ∇f

Divergence ….how to write it?

Consider a vector $\vec{F}$: can you construct a function $X(\vec{F})$ such that
$X(\vec{F})dV = \vec{F}.d\vec{S}$



Flux through BACK
$$f_B = -F_1 h_2 \delta u_2 h_3 \delta u_3$$

Flux through FRONT
$$f_F = F_1 h_2 \delta u_2 h_3 \delta u_3 + \frac{\partial}{\partial u_1}\left(F_1 h_2 \delta u_2 h_3 \delta u_3\right)\delta u_1$$

$$f_B + f_F = \left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right)\right]\delta u_1 \delta u_2 \delta u_3$$

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Divergence …

The LEFT + RIGHT pair gives

$$f_L + f_R = \left[\frac{\partial}{\partial u_2}\left(F_2 h_1 h_3\right)\right]\delta u_1 \delta u_2 \delta u_3$$

The BOTTOM + TOP pair gives

$$f_{Bottom} + f_{Top} = \left[\frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]\delta u_1 \delta u_2 \delta u_3$$

$$f_{TOTAL} = \left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(F_2 h_1 h_3\right) + \frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]\delta u_1 \delta u_2 \delta u_3$$

$$\frac{\vec{F}\,.\,\delta\vec{S}}{\delta V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(F_2 h_3 h_1\right) + \frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]$$

Now break a finite volume into small volume elements

Flux from neighbouring walls of two infinitesimal volume elements will cancel

Only faces which form the part of the boundary of the volume will not cancel

Divergence ….

This function is called DIVERGENCE, denoted by $\vec{\nabla}\,.\,\vec{F}$

$$\oiiint \vec{\nabla}\,.\,\vec{F}\,dV = \oiint \vec{F}\,.\,d\vec{S}$$

Called Gauss's theorem

"divergence" should convey a visual picture of the Vector field.... What is it?

How should a vector field look around points of stable/unstable equilibrium ?

Divergence of a vector is a scalar quantity

In Cartesian:

$$\vec{\nabla}\,.\,\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

In Spherical polar:

$$\vec{\nabla}\,.\,\vec{F} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\, F_r\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta\, F_\theta\right) + \frac{\partial}{\partial \phi}\left(r F_\phi\right)\right]$$

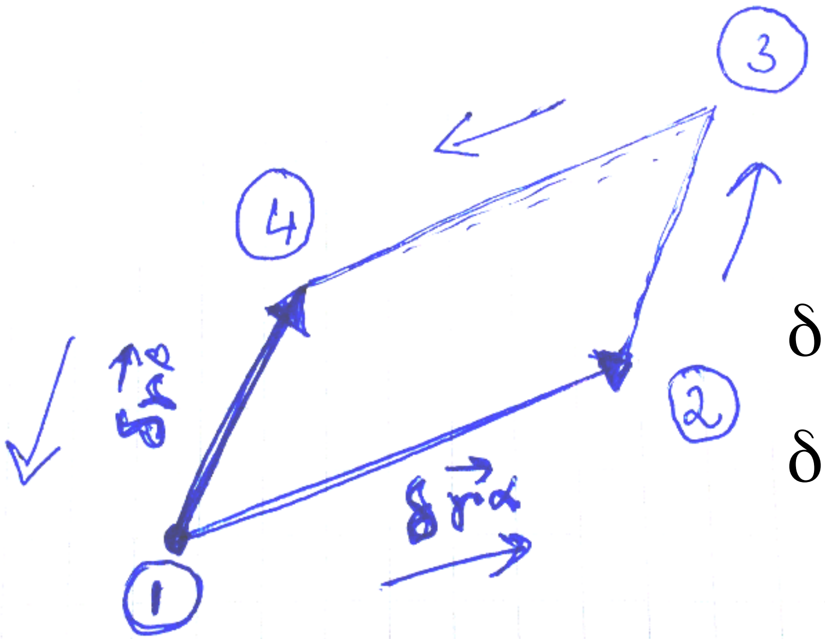
In cylindrical polar

$$\vec{\nabla}\,.\,\vec{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{\partial}{\partial \phi}\left(F_\phi\right) + \frac{\partial}{\partial z}\left(\rho F_z\right)\right]$$

Divergence and continuity equation....

Curl

Consider two arbitray infinitesimal displacements



$$\delta \vec{r}^{\alpha} = \hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 \delta u_1^{\alpha} + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 \delta u_2^{\alpha} + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 \delta u_3^{\alpha}$$

$$\delta \vec{r}^{\beta} = \hat{\boldsymbol{\epsilon}}_{\mathbf{1}} h_1 \delta u_1^{\beta} + \hat{\boldsymbol{\epsilon}}_{\mathbf{2}} h_2 \delta u_2^{\beta} + \hat{\boldsymbol{\epsilon}}_{\mathbf{3}} h_3 \delta u_3^{\beta}$$

The vector field is $\vec{F}$: Is it possible to have a function $X\left(\vec{F}\right)$ such that

$$X(\vec{F}).\delta\vec{S} = \sum_{\substack{peri-\\meter}} \vec{F}.\delta\vec{l}$$

Connect some characteristics of inside points with the boundary.

Curl

$$d\vec{S} = \delta\vec{r}^{\alpha} \times \delta\vec{r}^{\beta} = \begin{vmatrix} \hat{\boldsymbol{\epsilon}}_1 & \hat{\boldsymbol{\epsilon}}_2 & \hat{\boldsymbol{\epsilon}}_3 \\ h_1 \delta u_1^{\alpha} & h_2 \delta u_2^{\alpha} & h_3 \delta u_3^{\alpha} \\ h_1 \delta u_1^{\beta} & h_2 \delta u_2^{\beta} & h_3 \delta u_3^{\beta} \end{vmatrix}$$

$$\begin{aligned} X(\vec{F}).d\vec{S} = \; & X_1 h_2 h_3 [\delta u_2^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_2^{\beta}] \\ & - X_2 h_1 h_3 [\delta u_1^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_1^{\beta}] \\ & + X_3 h_1 h_2 [\delta u_1^{\alpha} \delta u_2^{\beta} - \delta u_2^{\alpha} \delta u_1^{\beta}] \end{aligned}$$

Try writing RHS in this form and compare.
The co-efficients of the arbitrary displacments must agree

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Curl

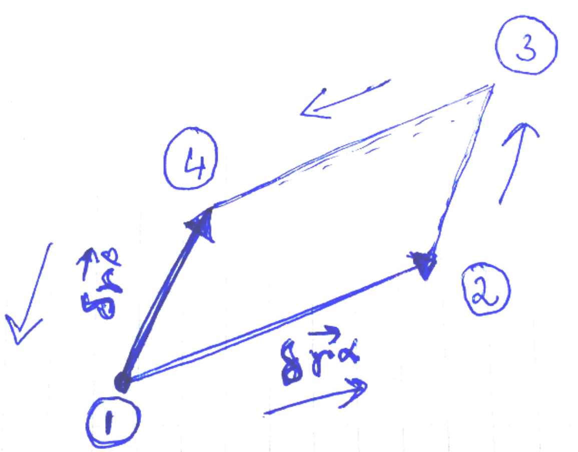
Consider the pair of paths $(1 \rightarrow 2)$ and $(3 \rightarrow 4)$

$$\vec{F}.\delta\vec{l}_{|1\rightarrow 2} = F_1 h_1 \delta u_1^{\alpha} + F_2 h_2 \delta u_2^{\alpha} + F_3 h_3 \delta u_3^{\alpha}$$

$$\vec{F}.\delta\vec{l}_{|3\rightarrow 4} = \left[F_i h_i + (\nabla F_i h_i).\delta\vec{r}^{\beta}\right](-\delta u_i^{\alpha}) \qquad (i=1,2,3)$$

Write contributions from $\vec{F}.\delta\vec{l}_{|2\rightarrow 3}$ & $\vec{F}.\delta\vec{l}_{|4\rightarrow 1}$ similarly.

Full path gives: $\left(\nabla\vec{F}.\delta\vec{r}^{\beta}\right).\delta\vec{r}^{\alpha} - \left(\nabla\vec{F}.\delta\vec{r}^{\alpha}\right).\delta\vec{r}^{\beta}$



$$= \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^{\beta}\right] h_k \delta u_k^{\alpha}$$

$$- \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^{\alpha}\right] h_k \delta u_k^{\beta}$$

$$= \sum_{k,i}\left[\frac{\partial F_i h_i}{\partial u_k} - \frac{\partial F_k h_k}{\partial u_i}\right]\delta u_i^{\beta}\delta u_k^{\alpha}$$

Work out the intermediate steps as an exercise

Curl

Now compare the co-efficient of $\delta u_2^{\alpha}\delta u_3^{\beta}-\delta u_3^{\alpha}\delta u_2^{\beta}$

We need to put $i=3, k=2$ and then $i=2, k=3$

this gives $X_1 h_2 h_3=\left[\frac{\partial F_3 h_3}{\partial u_2}-\frac{\partial F_2 h_2}{\partial u_3}\right]$

So $$X(\vec{F}) = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\boldsymbol{\epsilon}}_{\mathbf{1}} & h_2\hat{\boldsymbol{\epsilon}}_{\mathbf{2}} & h_3\hat{\boldsymbol{\epsilon}}_{\mathbf{3}} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \equiv \begin{cases} \nabla\times\vec{F} \\ curl\ \vec{F} \\ rot\ \vec{F} \end{cases}$$

We have $\iint \nabla\times\vec{F}\,.\,d\vec{S}=\oint \vec{F}\,.\,d\vec{l}$ $\quad(called\ Stoke's\ theorem)$

Now break a finite surface into small area elements

Line integral from neighbouring perimeters of two infinitesimal area elements will cancel

Only line segments which form the part of the perimeter will not cancel

Who needs electrostatics anyway ?

SEM, electron optics, mass-spectrometer

How well can you "see" the nano-world?

How well can measure & identify masses/ion-fragments?

Magnetic field also satisfies the laplacian... etc

Del^2 phi =0 appears in many places

Current flow in a conductor : why is it a similar problem?

Boundary value problems are everywhere.....

Note: there is no such thing as 2D electrostatics. 2D electrostatics means that the third co-ordinate can be droppped due to aspect ratio etc.

The Laplace equation

The mean value theorem
Poisson formula (for 2D boundary value problem)
Conformal mapping
Significance of the cylindrical co-ordinate
Off-axis expansion (electrostatic lensing)
Bessel functions

The mean value theorem

A scalar function $V(\vec{r})$ satisfies $\nabla^2 V = 0$

Consider a SPHERE of radius R : integrate $\nabla^2 V$ over the volume

$$\int_{vol} \vec{\nabla}.(\vec{\nabla} V)\, d\tau = \int_{surface} \vec{\nabla} V . d\vec{S}$$

Write the gradient in spherical polar

$$= \int [\hat{\epsilon}_r \frac{\partial V}{\partial r} + \hat{\epsilon}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\epsilon}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}] . d\vec{S}$$

$$= \int \frac{\partial V}{\partial r} R^2 \sin\theta\, d\theta\, d\phi$$

Only the radial component survives because d**S** points radially outwards

$$0 = R^2 \frac{\partial}{\partial r} \int_{surface} V(r, \theta, \phi) \sin\theta\, d\theta\, d\phi$$

The average value $\langle V(\theta, \phi) \rangle_r$ over a sphere is independent of r.

In the limit $r \to 0$, we must have $\langle V \rangle = V(0)$

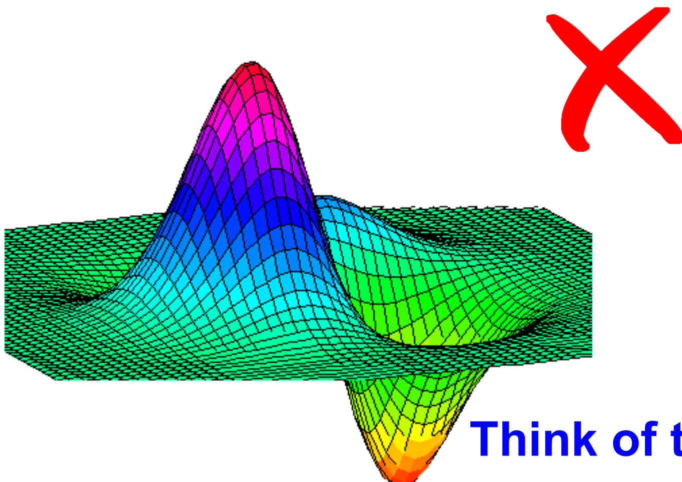
So average value over a spherical surface $=$ value at the center

In 2D one can do BETTER than this...we will see soon.

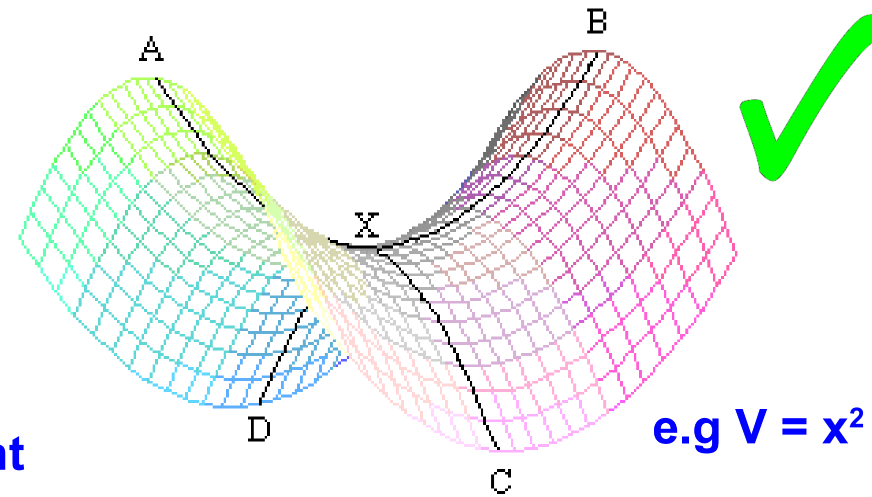
An obvious consequence

There are no maxima or minima of V in a region where $\nabla^2 V = 0$

But there can be saddle points



Think of the gradient near maxima/minima



e.g $V = x^2 - y^2$

No stable equilibrium possible in purely electrostatic field (*Earnshaw*)
All extremal values must occur at the boundary

$V = const$ on ALL points on ALL boundaries $\Rightarrow V$ is constant everywhere

UNIQUENESS: There is only one possible solution of $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$
consistent with a given boundary condition

The 2D polar solution → Poisson formula

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

Try: $V = R(r)e^{im\theta}$

Why not $e^{m\theta}$?

This gives:

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} - m^2 R = 0$$

trial solution $R = Ar^n$ gives : $n = \pm m$, so

$$V(r,\theta) = \sum_m \left(A_m r^m + \frac{B_m}{r^m}\right) e^{\pm i m\theta}$$

Special case $m = 0$: $R = A_0 + B_0 \ln r$

Full soln : $V(r,\theta) = (A_0 + B_0 \ln r) + \sum_{m\neq 0}\left(A_m r^m + \frac{B_m}{r^m}\right)e^{\pm i m\theta}$

The Poisson formula

The potential is $f(\theta)$ on the unit circle.
Find the potential everywhere inside.

$$\nabla^2 \phi = 0$$
$$\phi(r,\theta) = ?$$

$$V(r,\theta) = \sum_{-\infty}^{\infty} A_m r^{|m|} e^{i m \theta}$$

$$A_m = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) e^{-i m \alpha} d\alpha$$

$$V = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_0^{2\pi} f(\alpha) e^{-i m \alpha} r^{|m|} e^{i m \theta} d\alpha$$

Interchange integration and summation
Add up the geometric series first....

The Poisson formula

$$V(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, f(\alpha)\left(\sum_{m=-\infty}^{m=0} r^{|m|}e^{im(\alpha-\theta)} + \sum_{m=0}^{m=\infty} r^{|m|}e^{im(\alpha-\theta)} - 1\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, f(\alpha)\left(\frac{1}{1 - r e^{-i(\alpha-\theta)}} + \frac{1}{1 - r e^{i(\alpha-\theta)}} - 1\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, f(\alpha)\left(\frac{2 - 2r\cos(\theta-\alpha)}{1-2r\cos(\theta-\alpha)+r^2} - 1\right)$$

$$\boxed{V(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, f(\alpha)\left(\frac{1 - r^2}{1-2r\cos(\theta-\alpha)+r^2}\right)}$$

A similar relation can be derived for $r > 1$

Using complex numbers

$$\left.\begin{aligned} u+iv &= f(x+iy) \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \Rightarrow \begin{aligned} \nabla^2 u(x,y) &= 0 \\ \nabla^2 v(x,y) &= 0 \\ (\nabla u).(\nabla v) &= 0 \end{aligned}$$

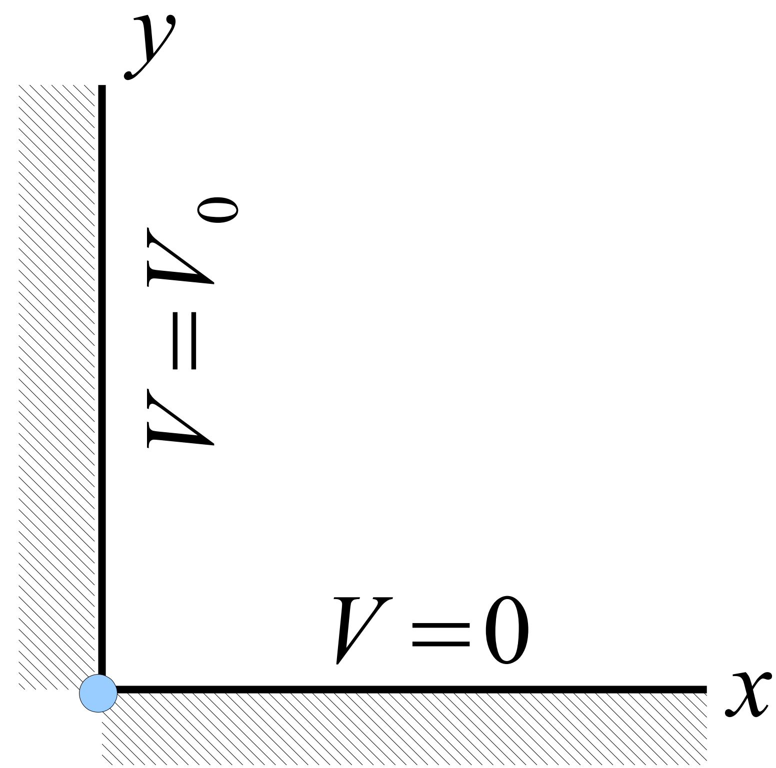
Lines of $u = const.$ and $v = const$ are normal to each other

Make guesses, visualize some function. If the boundary conditions match, uniqueness gurantees you have the solution.

It is useful to remember the "equipotential contours" of some functions...straight lines, circles, ellipses, hyperbola etc

The potential given on a "wedge"

$$F(z) = \frac{2V_0}{\pi}\ln z$$
$$v(x,0) = 0$$
$$v(0,y) = \frac{2V_0}{\pi}\frac{\pi}{2}$$
$$v(x,y) = \frac{2V_0}{\pi}\arctan\left(\frac{y}{x}\right)$$



The solution satisfies Laplace's eqn & boundary conditions. So it must be the unique solution.

Modify the solution for an arbitrary angle between the two sides .

Question : How does the electric field "lines of force" look ?

Elliptical and hyperbolic equipotentials

$$\boxed{W \equiv u + iv = \cosh^{-1} z}$$

$$x = \cosh u \ \cos v$$

$$y = \sinh u \ \sin v$$

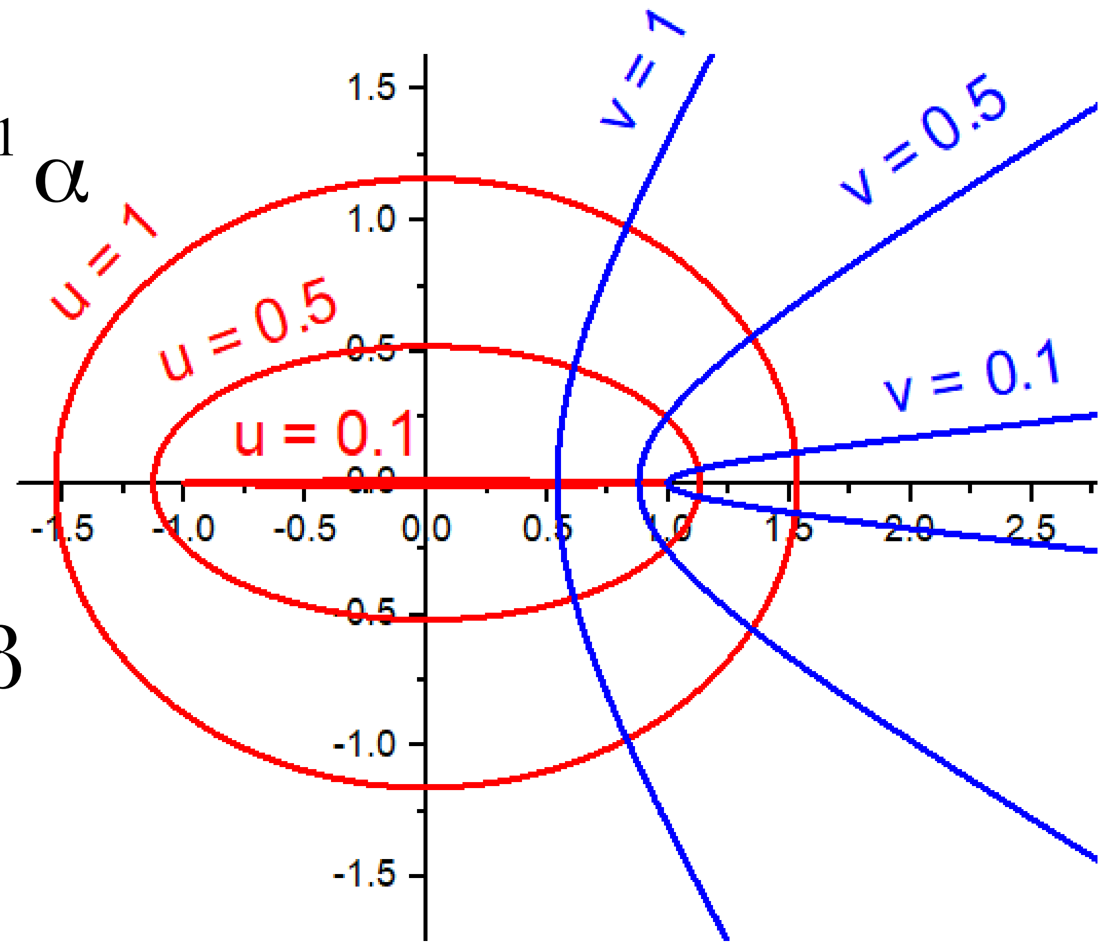
$$\text{if } u = \text{const} = \cosh^{-1}\alpha$$

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\alpha^2 - 1} = 1$$

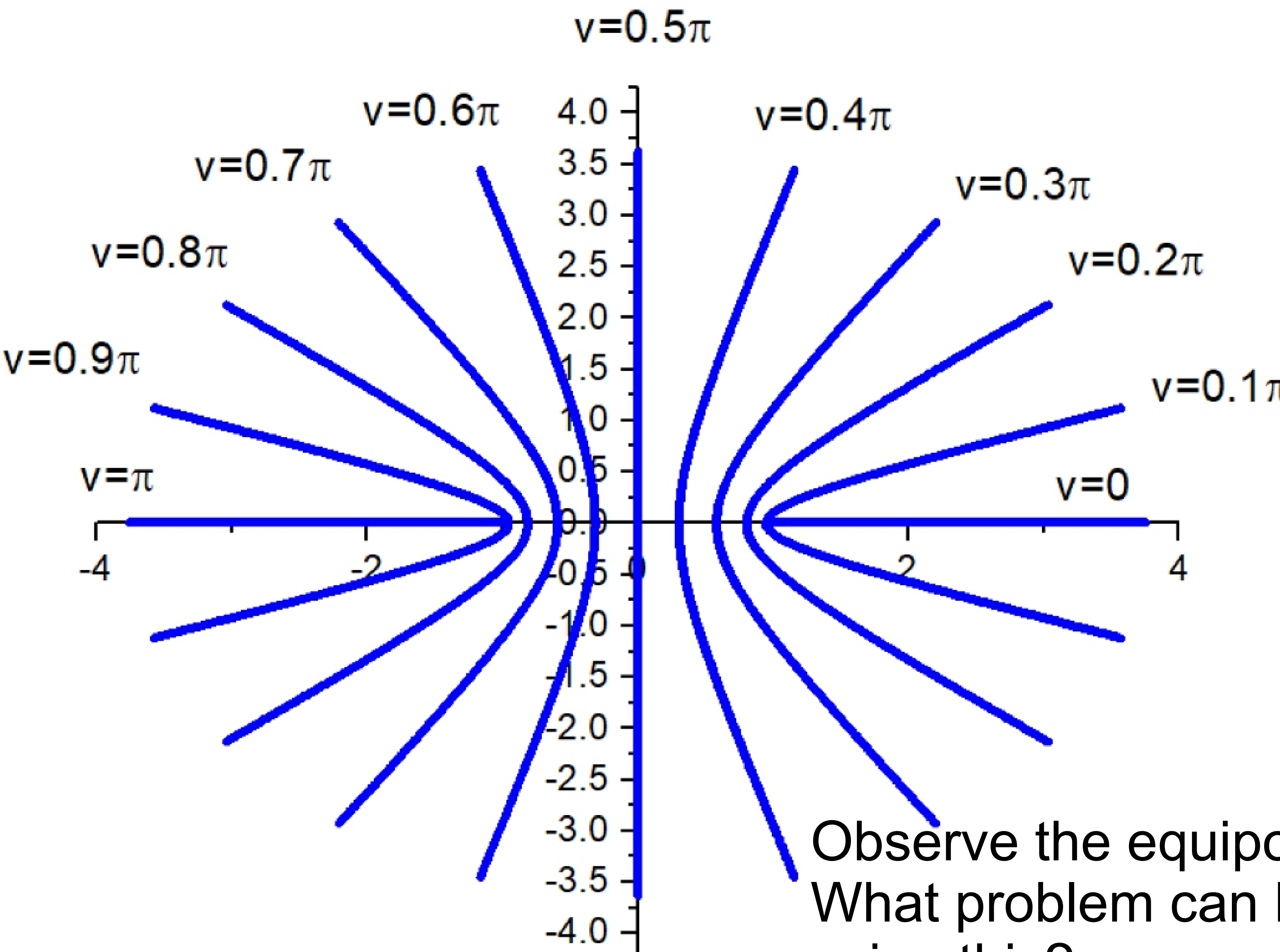
These are CONFOCAL.

$$\text{if } v = \text{const} = \cos^{-1}\beta$$

$$\frac{x^2}{\beta^2} - \frac{y^2}{1 - \beta^2} = 1$$

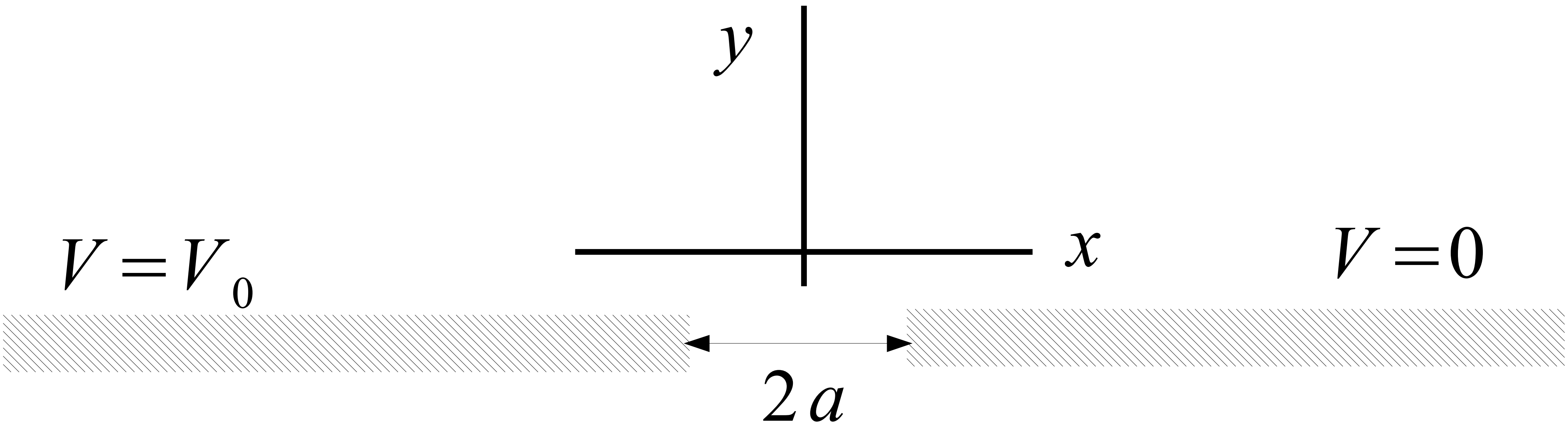


A slit and hyperbolic equipotentials



Observe the equipotentials. What problem can be solved using this?

How do we scale the variables ?



The slit width $= 2a$ extends in the xz-plane

$V(x, y)$ should satisfy

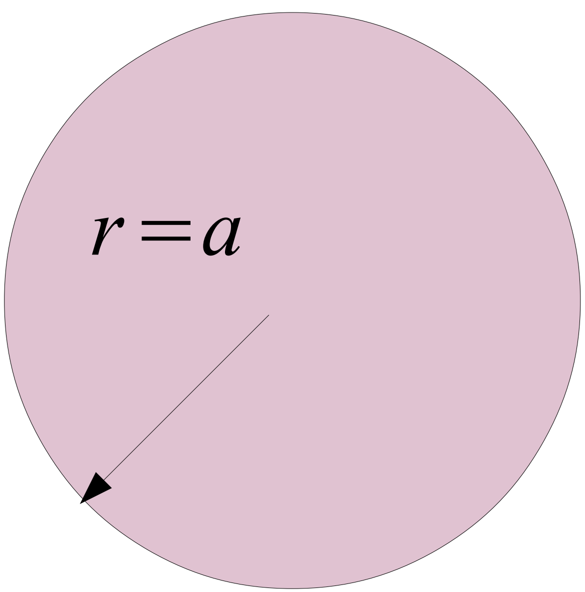
$$\frac{x^2}{\cos^2 \frac{\pi V}{V_0}} - \frac{y^2}{\sin^2 \frac{\pi V}{V_0}} = a^2$$

Design your co-ordinate to suit a problem

Easy problem: A long copper pipe (circular cross section, radius a) is kept at a potential V. What is the electric field everywhere?

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

$$\text{Full soln} \quad : \quad V(r,\theta) = (A_0 + B_0 \ln r) \; + \; \sum_{m\neq 0}\left(A_m r^m + \frac{B_m}{r^m}\right)e^{\pm i m\theta}$$



$$V(a,\theta) \quad = \quad V_0 \;\; \forall \;\; \theta$$

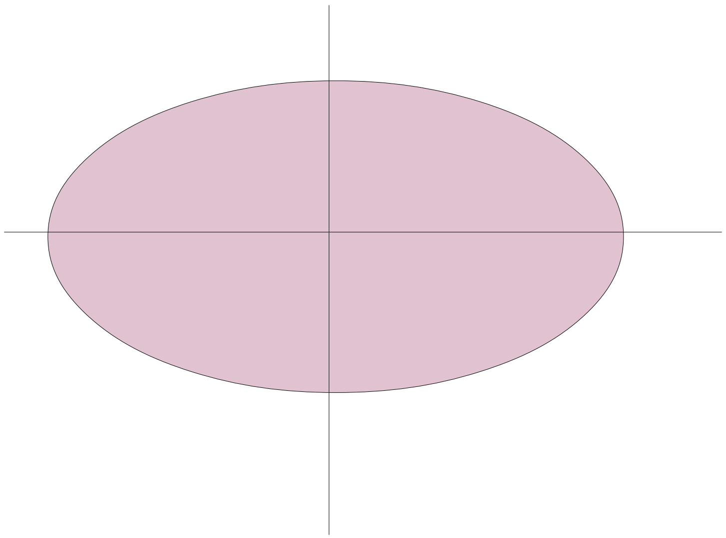
$$V(r,\theta) \quad = \quad V_0\left(1+\ln\frac{a}{r}\right)$$

$$\therefore E_r \quad = \quad \frac{V_0}{r} \qquad (r>a)$$

$$E_\theta \quad = \quad 0$$

Design your co-ordinate to suit a problem

Not so easy problem: A long copper pipe (elliptical cross section) is kept at a potential V. What is the electric field everywhere?



$$\boxed{\frac{x^2}{2}+y^2=1}$$

Strategy : Design/find a co-ordinate system (u,v) in which u=constant or v= constant produces an ellipse.

See if Laplacian is separable in that (u,v) co-ordinate system.

Solve Laplacian, now you should get a tractable boundary value problem. It will not work in all cases...but in some cases.

The elliptical co-ordinate (u, v, z)

$$\left.\begin{aligned} \boxed{\begin{aligned} x &= \cosh u \cos v \\ y &= \sinh u \sin v \end{aligned}} \\ z &= z \end{aligned}\right\} \Rightarrow h_u = h_v = \sqrt{\sinh^2 u + \sin^2 v}$$

$$\nabla . \vec{F} = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}\left(F_u h_v\right) + \frac{\partial}{\partial v}\left(F_v h_u\right)\right]$$

$$\nabla^2 V = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}\left(\frac{1}{h_u}\frac{\partial V}{\partial u} h_v\right) + \frac{\partial}{\partial v}\left(\frac{1}{h_v}\frac{\partial V}{\partial v} h_u\right)\right]$$

$$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = 0 \quad (\text{a fortuitous case !!})$$

This means we can trivially write down the solution in (u,v)
By recalling the solution in simple cartesian (x,y)

The elliptical co-ordinate (u, v, z)

$$V(u, v) = (A_0 u + B_0)(C_0 v + D_0) + \sum_{k=1}^{\infty} (A_k \cosh ku + B_k \sinh ku)(C_k \sin kv + D_k \cos kv)$$

$u = \cosh^{-1}\sqrt{2}$ reproduces the required ellipse

$$V(\cosh^{-1}\sqrt{2}, v) = V_0 \quad \forall \quad v$$

$$\Rightarrow V(x, y) = \frac{V_0}{\cosh^{-1}\sqrt{2}} u$$

We need to invert : $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\cosh^2 u - 1} = 1$

The elliptical co-ordinate (u, v, z)

$$
\begin{aligned}
\cosh^2 u &= \frac{(x^2+y^2+1) \pm \sqrt{\left[(x-1)^2+y^2\right]\left[(x+1)^2+y^2\right]}}{2} \\
&\equiv \lambda(x, y) \\
V(x, y) &= \frac{V_0}{\cosh^{-1}\sqrt{2}} \cosh^{-1}\sqrt{\lambda(x, y)}
\end{aligned}
$$

$$
\begin{aligned}
E_x(x, y) &= \frac{V_0}{2\cosh^{-1}\sqrt{2}} \frac{1}{\sqrt{\lambda(\lambda-1)}} \frac{\partial \lambda}{\partial x} \\
E_y(x, y) &= \frac{V_0}{2\cosh^{-1}\sqrt{2}} \frac{1}{\sqrt{\lambda(\lambda-1)}} \frac{\partial \lambda}{\partial y}
\end{aligned}
$$

Calculate the limiting forms for large x,y and fix the sign. Show that your recover the result for the circular pipe as expected..

What is meant by "conformal"

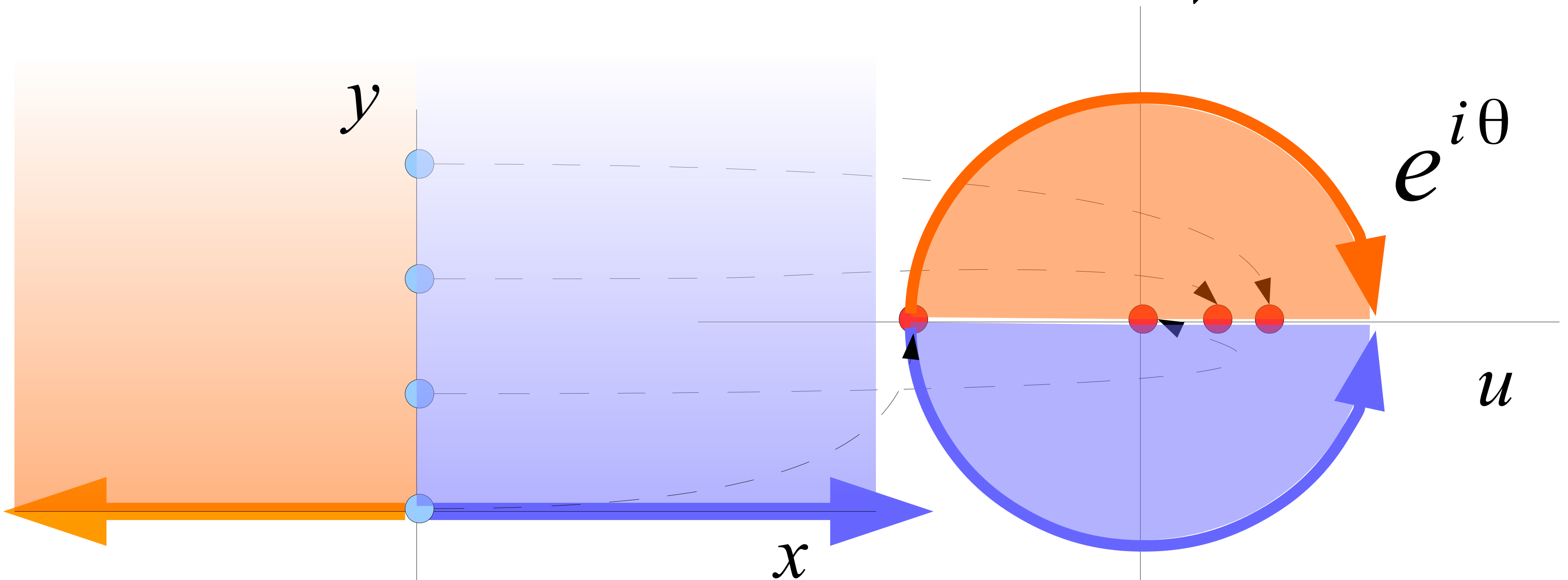
The Z and W plane.

Angle between two trajectories at their point of intersection

Possibility of generating many orthogonal co-ordinates starting from cartesian

Conformal map : upper half plane to unit circle

$$W \equiv u + iv = \frac{z-i}{z+i} \qquad (y \geq 0)$$

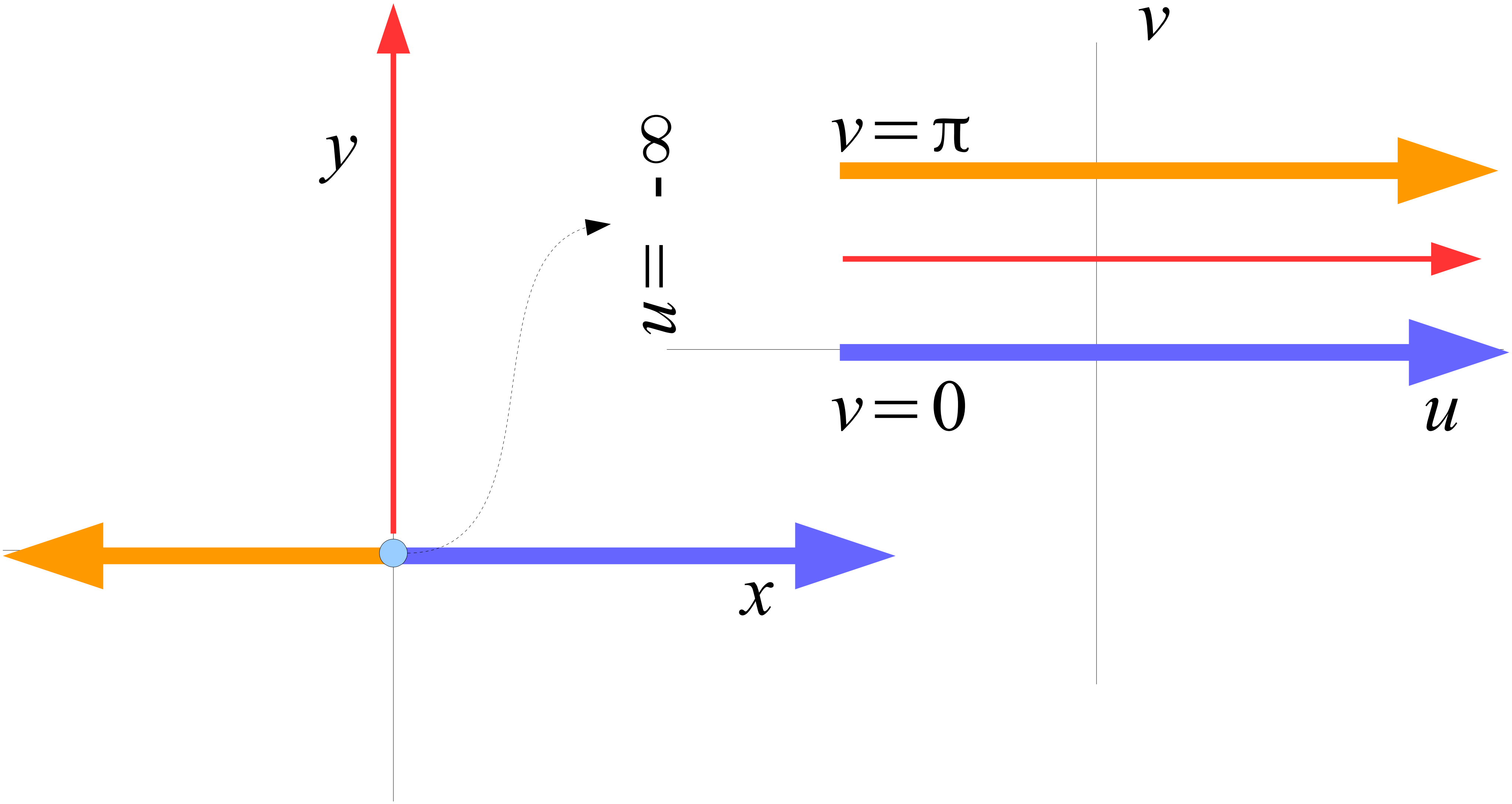


$$\boxed{\frac{z-z_0}{z+\bar{z}_0} \Rightarrow ?}$$

$$-\frac{2x}{x^2-1} = \tan\theta \quad (\text{on real line})$$

Conformal map : upper half plane to a strip

$$W \equiv u + iv = \ln z \qquad (y > 0)$$



How to use this ? The key fact.

We have a function of $W = u + i\,v = f\,(x + i\,y)$

And a function $\Phi(u, v)$ such that

$$\frac{\partial^2 \Phi(u, v)}{\partial u^2} + \frac{\partial^2 \Phi(u, v)}{\partial v^2} = 0$$

now since we have

$$\begin{aligned} W &= f(z) \\ \Phi(u, v) &= \Phi(u(x, y), v(x, y)) \\ &\equiv \psi(x, y) \end{aligned}$$

$$\Rightarrow \frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2} = 0$$

Notice the variables. !!! It is NOT a trivial assertion !!!

Conformal map the algorithm for using it

Suppose we know how to solve a problem with a given boundary condition in the (u,v) plane.

Then, if we can find a "conformal map", that twists the boundary from the W-plane to a desired boundary in the z-plane. The variables are now (x,y)

And somehow one of these two boundaries is "simpler" and the integral can be done exactly.

The theorem ensures that one solution can be exactly mapped into the other. Uniqueness gurantees that is the correct solution.

Remember : There is NO set recipe for finding the correct map !!!

Why does the method work ?

$$\psi(x,y) \equiv \Phi(u(x,y),v(x,y))$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \Phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \Phi}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left[\frac{\partial}{\partial u}\left(\frac{\partial \Phi}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial \Phi}{\partial u}\right)\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial x} + \frac{\partial \Phi}{\partial u}\frac{\partial^2 u}{\partial x^2}$$

$$+\left[\frac{\partial}{\partial u}\left(\frac{\partial \Phi}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial \Phi}{\partial v}\right)\frac{\partial v}{\partial x}\right]\frac{\partial v}{\partial x} + \frac{\partial \Phi}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 \Phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 \Phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + 2\frac{\partial^2 \Phi}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial \Phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial \Phi}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Φ is harmonic in $(u, v) \rightarrow \Psi$ is harmonic in (x, y)

Similarly....

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 \Phi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial^2 \Phi}{\partial u \partial v}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)$$

$$+ \frac{\partial \Phi}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial \Phi}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$\boxed{\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}}$$

Adding the two & using the Cauchy-Riemann relations...

$$\nabla^2 \psi = \underbrace{\nabla^2 \Phi}_{=0}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2\frac{\partial^2 \Phi}{\partial u \partial v}\underbrace{(\nabla u).(\nabla v)}_{=0}$$

$$+ \frac{\partial \Phi}{\partial u}\underbrace{\nabla^2 u}_{=0} + \frac{\partial \Phi}{\partial v}\underbrace{\nabla^2 v}_{=0}$$

$$= 0$$

Using the conformal map

Suppose the potential is known on the x-axis.
Given $\phi(\lambda, 0)$, How can you find $\phi(x, y)$?
Strategy: Transform the real line unit circle
Ensure that (x, y) is the center of the circle.

STEP 1: The transformation...

$$W = \frac{z - z_0}{z - \bar{z}_0} \qquad \text{where} \qquad z_0 \equiv x + iy$$

STEP 2: What happens to any point $(\lambda, 0)$ the real line ?

$$e^{i\theta} = \frac{\lambda - (x + iy)}{\lambda - (x - iy)} \quad : \text{ define } \quad f(\theta) = \phi(\lambda, 0)$$

Using the conformal map

STEP 3 : relate $$d\theta = \frac{2y}{(\lambda - x)^2 + y^2} d\lambda$$

STEP 4

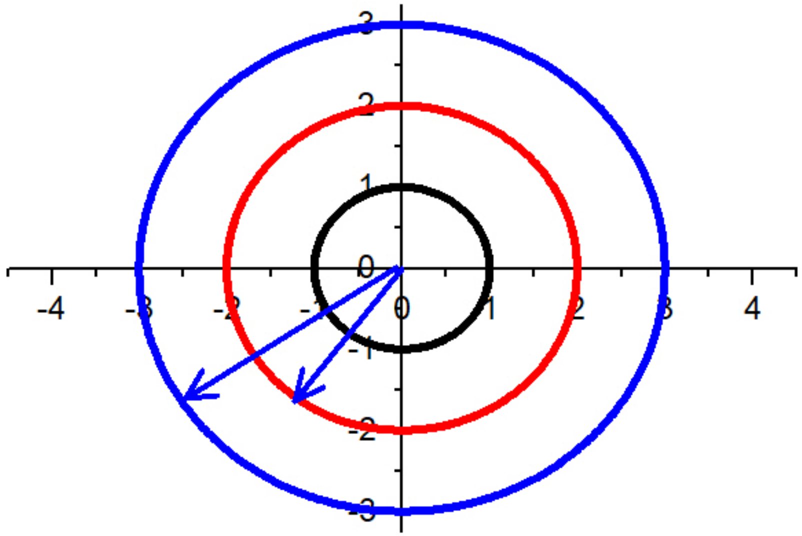
$$\phi(x,y) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

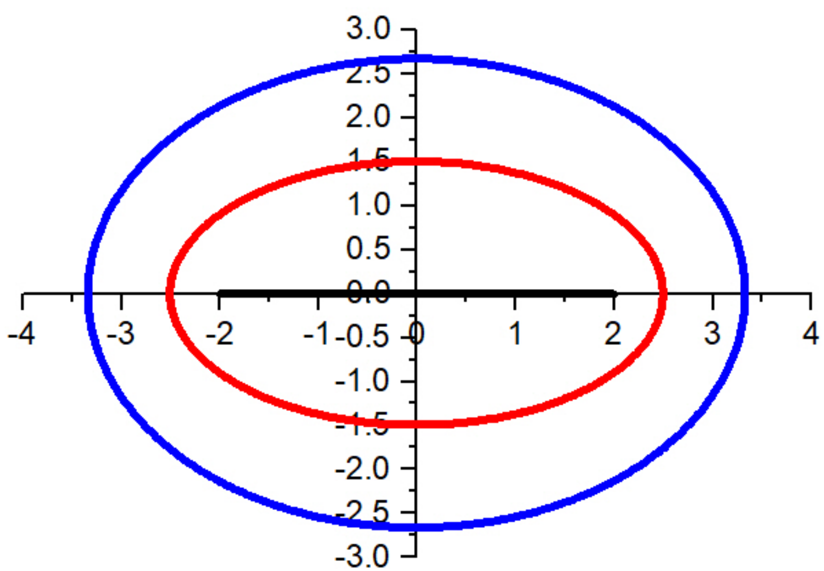
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(z-\bar{z}_0)\phi(\lambda,0)}{(\lambda - z_0)(\lambda - \bar{z}_0)} d\lambda$$

$$\boxed{\phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,\phi(\lambda,0)}{(x-\lambda)^2 + y^2} d\lambda}$$

A closed form expression in terms of the boundary values.
But it will NOT help in solving for slits, apertures etc

A more complex map : the Jukowski map





What happens to

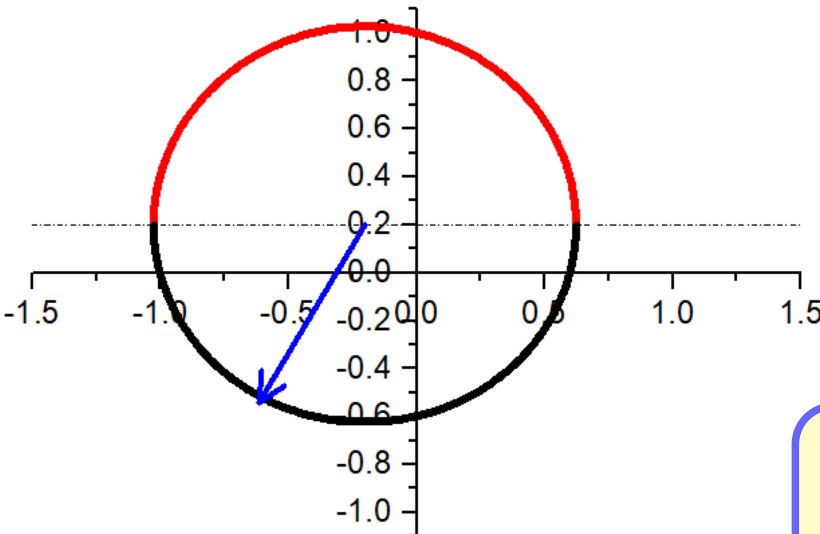
$$|z| = 1$$
$$|z| = 2$$
$$|z| = 3$$

when transformed by

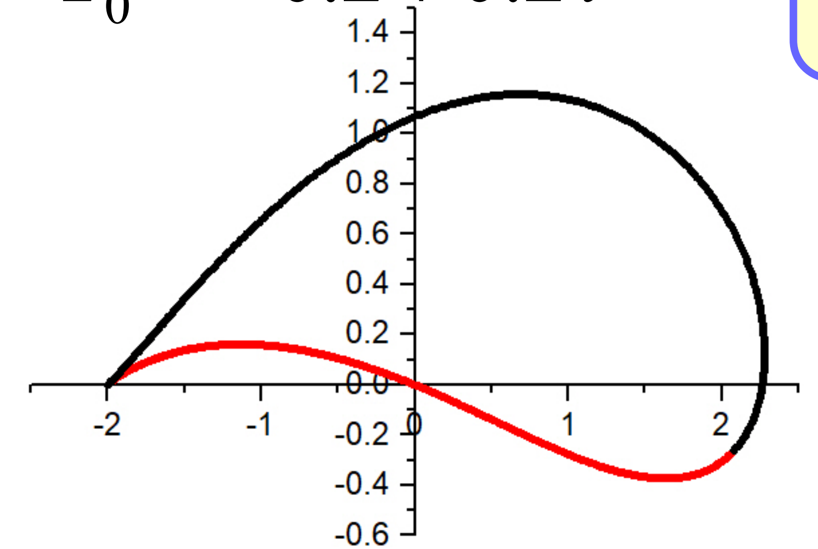
$$W = z + \frac{1}{z}$$

Deforming circles to straight line and confocal ellipses

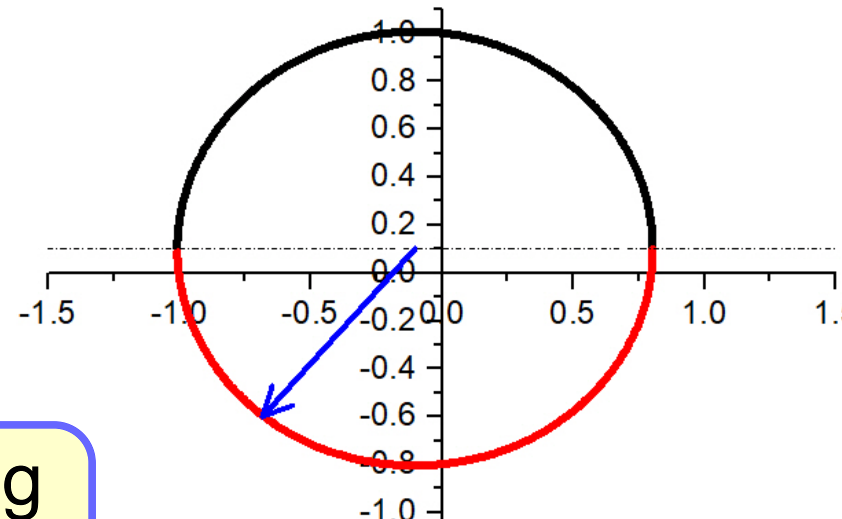
Transforming $|z - z_0| = |1 - z_0|$



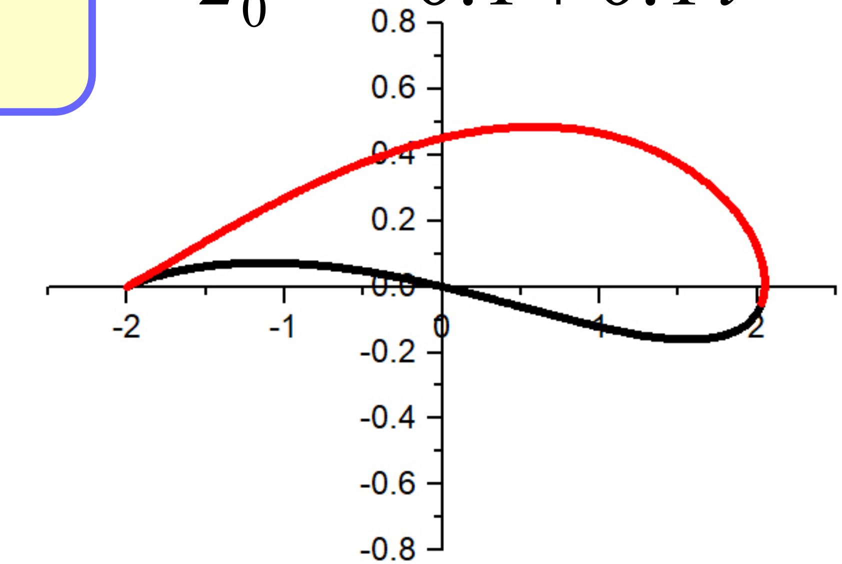
$$z_0 = -0.2 + 0.2\,i$$



Deforming circles to aerofoil



$$z_0 = -0.1 + 0.1\,i$$



Cylindrical co-ordinate system

Off-axis expansion
How electrostatic lensing works
Bessel functions

Potentials with axial symmetry

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If the beam does not change the potential

$$\frac{\partial V}{\partial \rho} + \rho \frac{\partial^2 V}{\partial \rho^2} + \rho \frac{\partial^2 V}{\partial z^2} = 0$$

Axially symmetric

If V(0,z) is known the complete potential & trajectory can be determined.

First solve a generic problem for axially symmetric solution of laplace eqn

$$\frac{\partial V}{\partial \rho} + \rho \frac{\partial^2 V}{\partial \rho^2} + \rho \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(\rho, z) = \sum_{n=0}^{\infty} A_{2n}(z)\rho^{2n}$$

$$V(0,z) = A_0(z)$$

Can couple even powers to even powers only.
Consider the powers of ρ.
First & second term will reduce power by 1.
Third term increases the power by 1.
No coupling between ρ^n and ρ^{n+1} possible.

$$\sum_{n=1}^{\infty} A_{2n}(z).2n.\rho^{2n-1} + \sum_{n=1}^{\infty} A_{2n}(z).2n.(2n-1).\rho^{2n-1} + \sum_{n=0}^{\infty}\left(\frac{d^2}{dz^2}A_{2n}(z)\right)\rho^{2n+1}$$

Potentials with axial symmetry : E_r and E_z

$$\sum_{n=1}^{\infty} A_{2n}(z).2n.\rho^{2n-1}+\sum_{n=1}^{\infty} A_{2n}(z).2n.(2n-1).\rho^{2n-1}+\sum_{n=0}^{\infty}\left(\frac{d^2}{dz^2}A_{2n}(z)\right)\rho^{2n+1}$$

Consider the coefficient of ρ

$$A_2(2+2.1)+A_0''(z) = 0 \quad \Rightarrow \quad A_2 = -\frac{A_0''}{4}$$

Consider the coefficient of ρ^3

$$A_4(4+4.3)+A_2''(z) = 0 \quad \Rightarrow \quad A_4 = \frac{A_0''''}{64}$$

Can you write the general term in the expansion ?

Try to find the pattern of the coefficients.

$$\frac{(-1)^n}{(n!)^2}\left(\frac{\rho}{2}\right)^{2n} A_0^{(2n)}(z)$$

The series solution is then :

$$V(\rho,z) = V(0,z) - \frac{V''(0,z)}{4}\rho^2 + \frac{V''''(0,z)}{64}\rho^4 - \ .....$$

$$E_r = -\frac{\partial V}{\partial \rho}=\frac{1}{2}\rho V''(0,z)$$

$$E_z = -\frac{\partial V(0,z)}{\partial z}=-V'(0,z)$$

Correct to first order
Terms of order ρ^2 and higher dropped

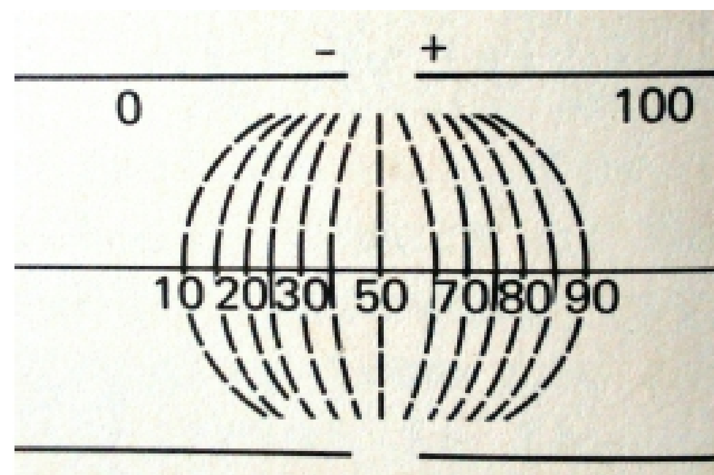
Electrons and light : Bethe's observation

optical refraction : $n_1 \sin\theta_1 = n_2 \sin\theta_2$

$v_{1\parallel} = v_{2\parallel}$

Choose $\phi = 0$ position, so that

$$\frac{mv^2}{2} + q\phi = 0 \quad \Rightarrow v \propto \sqrt{\phi}$$



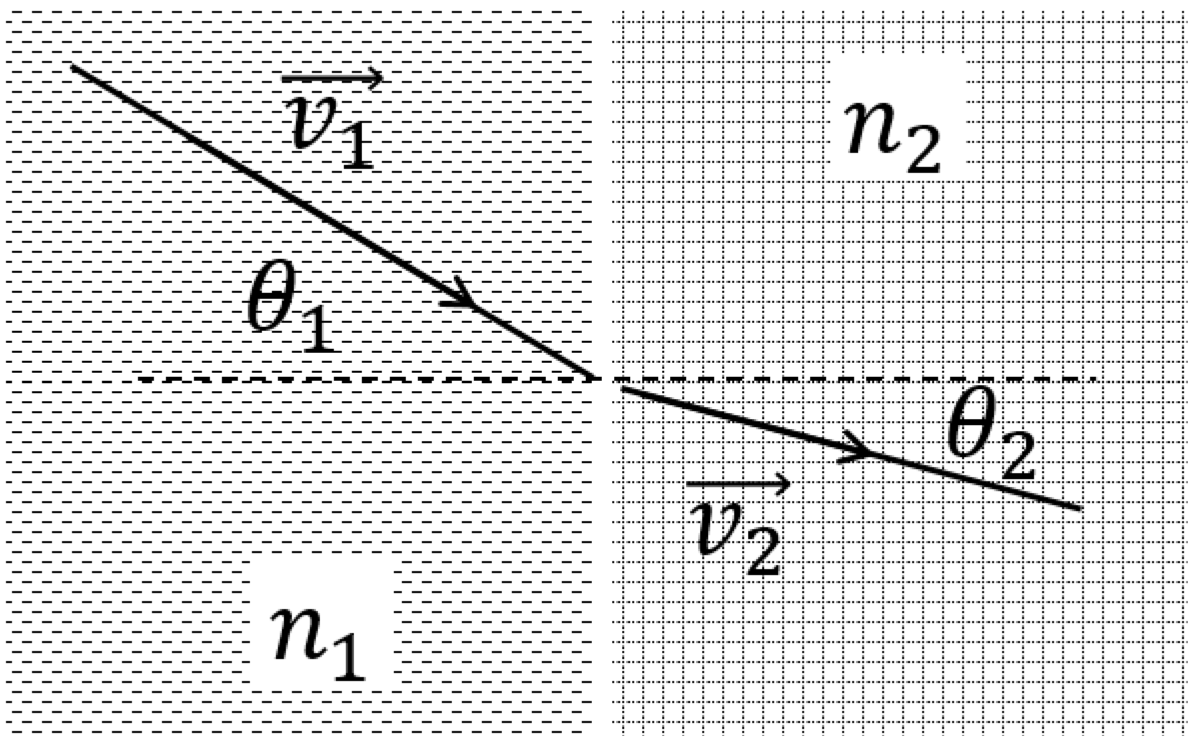
$$\frac{\sqrt{\phi_1}}{v_1} v_{1\parallel} = \frac{\sqrt{\phi_2}}{v_2} v_{2\parallel}$$

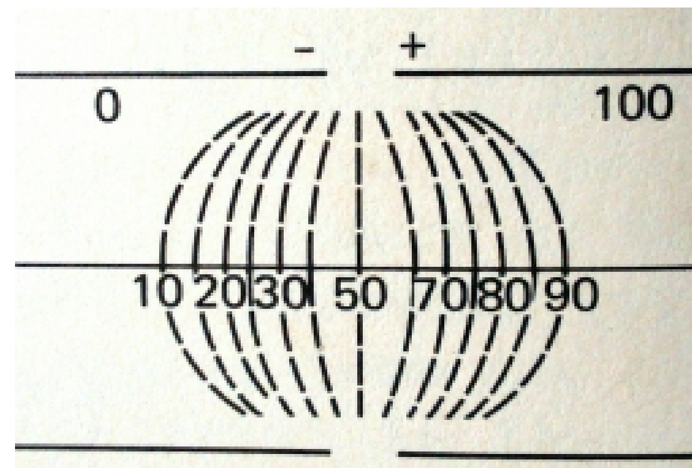
$$\sqrt{\phi_1} \sin\theta_1 = \sqrt{\phi_2} \sin\theta_2$$

square root of $\sqrt{\phi}$ $\rightarrow$ refractive index

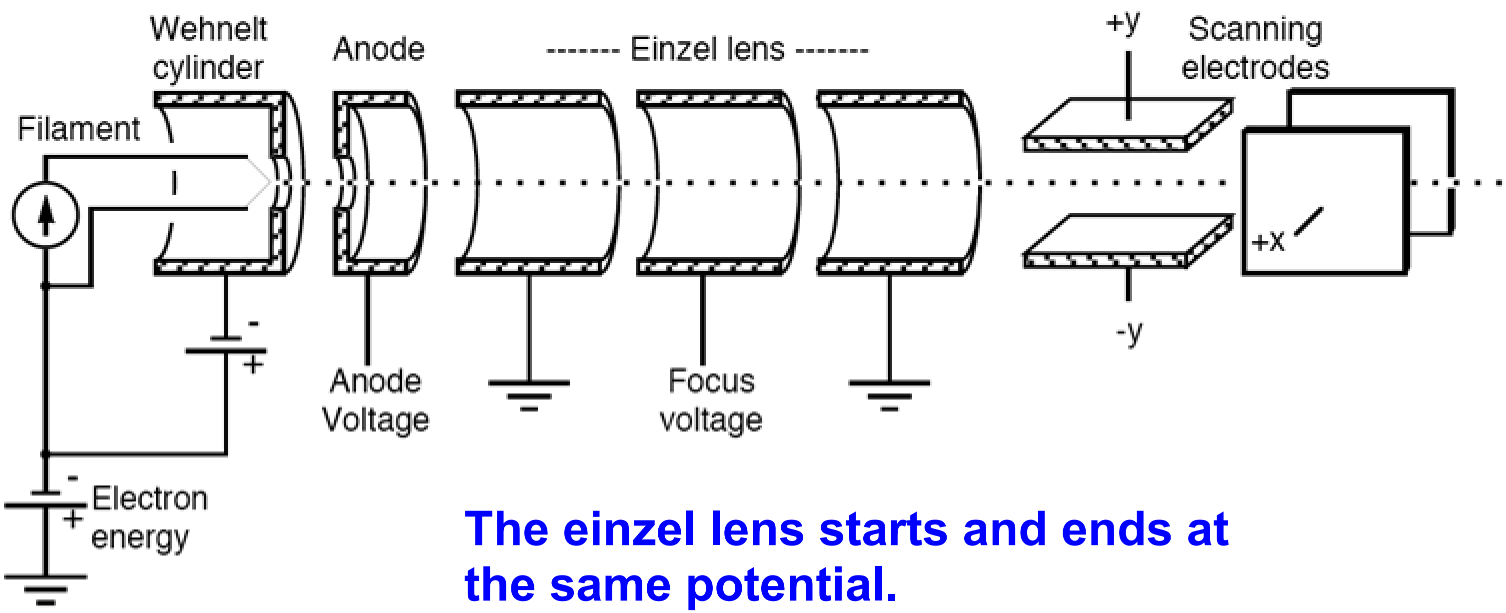
Electrons and light : Bethe's observation

optical refraction : $n_1 \sin\theta_1 = n_2 \sin\theta_2$

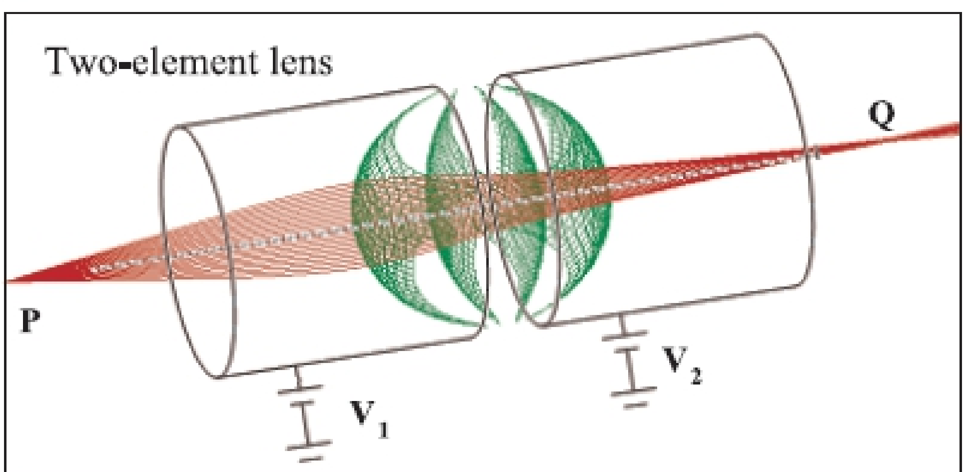


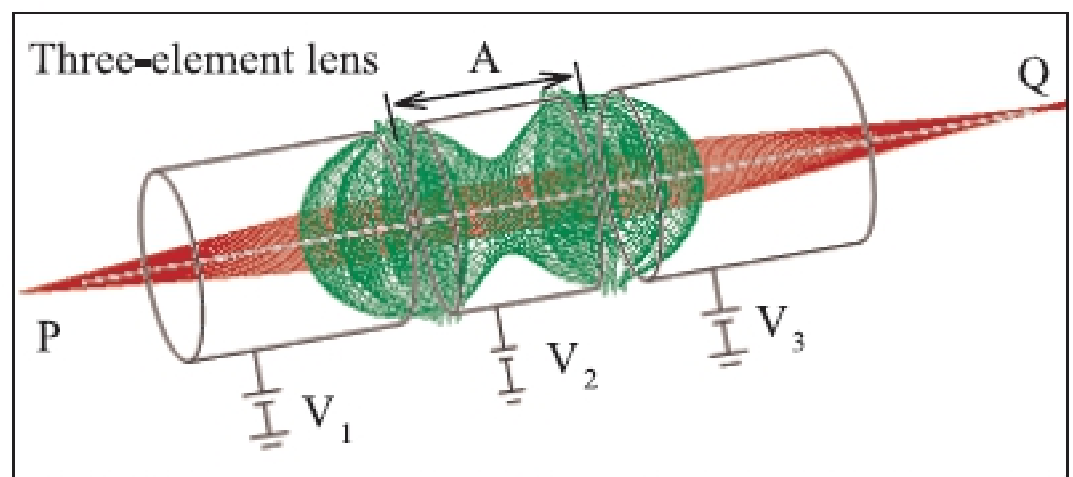


The einzel (= single) lens



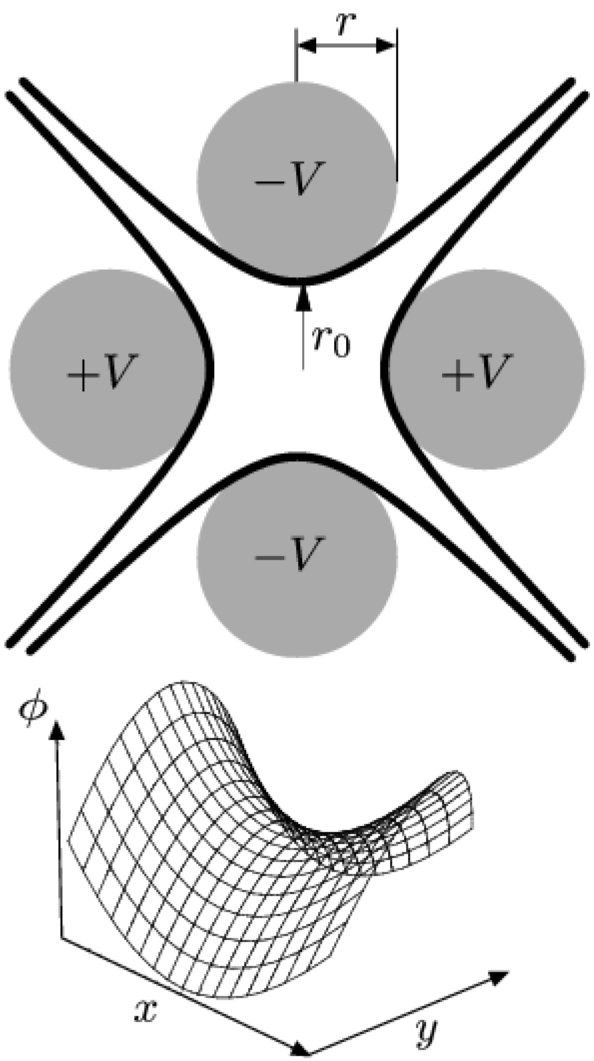
The einzel lens starts and ends at the same potential.

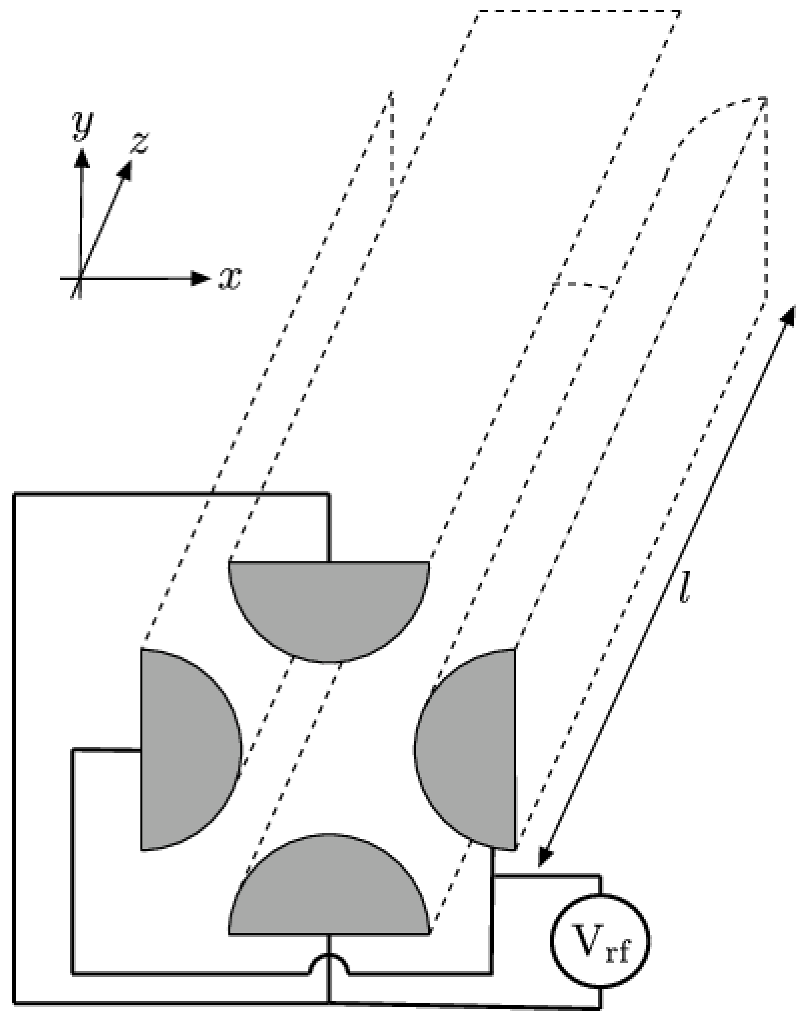




Equipotentials near gapped cylinders... Sise et al *Eur. J. Phys.* **29** (2008) 1165–1176

Quadrupole lens : NOT axially symmetric





Useful for correcting astigmatic error features in images.

Formal solution in (ρ, θ, z) co-ordinate

$$\nabla^2 V = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial V}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial V}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2} = 0$$

$$V(\rho, \theta, z) = R(\rho)\Phi(\theta)Z(z)$$

Separate out Φ

Separation of variables:
Standard method

$$\frac{1}{R}\frac{d^2 R}{d\rho^2}+\frac{1}{R\rho}\frac{dR}{d\rho}+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\theta^2}+\frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

$$\frac{\rho^2}{R}\frac{d^2 R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\frac{\rho^2}{Z}\frac{d^2 Z}{dz^2} = -\frac{1}{\Phi}\frac{d^2\Phi}{d\theta^2} = m^2$$

$$\Phi(\theta)=\Phi(\theta+2n\pi) \ : \ \therefore \Phi \sim e^{\pm i m\theta} \ : \ m=0,\pm 1,\pm 2....$$

Separation of variables in (ρ, θ, z)

Separate out $Z(z)$

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} - \frac{m^2}{\rho^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2$$

The sign of $k^2 \Rightarrow Z(z \to \infty) = 0$

The Radial equation :

$$\boxed{\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0}$$

If $m=0, k=0 \begin{cases} R = (A_0 + B_0 \ln\rho) \\ \Phi = const \\ Z = (C_0 + D_0 z) \end{cases}$

$\rho \to \infty$

$$\frac{d^2R}{d\rho^2} + k^2 R \approx 0$$

$\Rightarrow$ oscillation :

infinite polynomial

$\rho \to 0$

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + m^2 R \approx 0$$

$\Rightarrow R \sim \rho^{\pm m}$

Solving the radial equation in (ρ, θ, z)

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) R = 0 \quad (\text{with} \quad x = k\rho)$$

$$x^2 \frac{d^2 R}{dx^2} + x\frac{dR}{dx} + \left(x^2 - m^2\right) R = 0$$

$$R = x^m \sum_0^\infty a_j x^j \quad (x^{-m} \text{ not well behaved at } x=0)$$

co-efficient of x^m : a_0 arbitrary choice

co-efficient of x^{m+1} : $a_1(2m+1)=0$: $\Rightarrow a_1 = 0$

co-efficient of x^{m+2}

$$a_2\left[(2+m)^2 - m^2\right] = -a_0 \;:\; \Rightarrow a_2 = \frac{-1}{2.(2m+2)} a_0$$

Only alternate powers will be there in the series

Series solution of the radial equation

co-efficient of x^{m+4}

$$a_4\left[(4+m)^2-m^2\right]=-a_2 \quad : \quad \Rightarrow a_4=\frac{(-1)^2}{2.4.(2m+2)(2m+4)}a_0$$

co-efficient of x^{m+6}

$$a_6\left[(6+m)^2-m^2\right]=-a_4 \quad : \quad \Rightarrow a_6=\frac{(-1)^3}{2.4.6.(2m+2)(2m+4)(2m+6)}a_0$$

$$a_{2j}=(-1)^j\frac{m!}{2^{2j}j!(j+m)!}a_0 \;\rightarrow\; \frac{(-1)^j\Gamma(m+1)}{2^{2j}j!\Gamma(j+m+1)}a_0$$

Allow fractional values of m & choose $a_0=\dfrac{1}{2^m\Gamma(m+1)}$

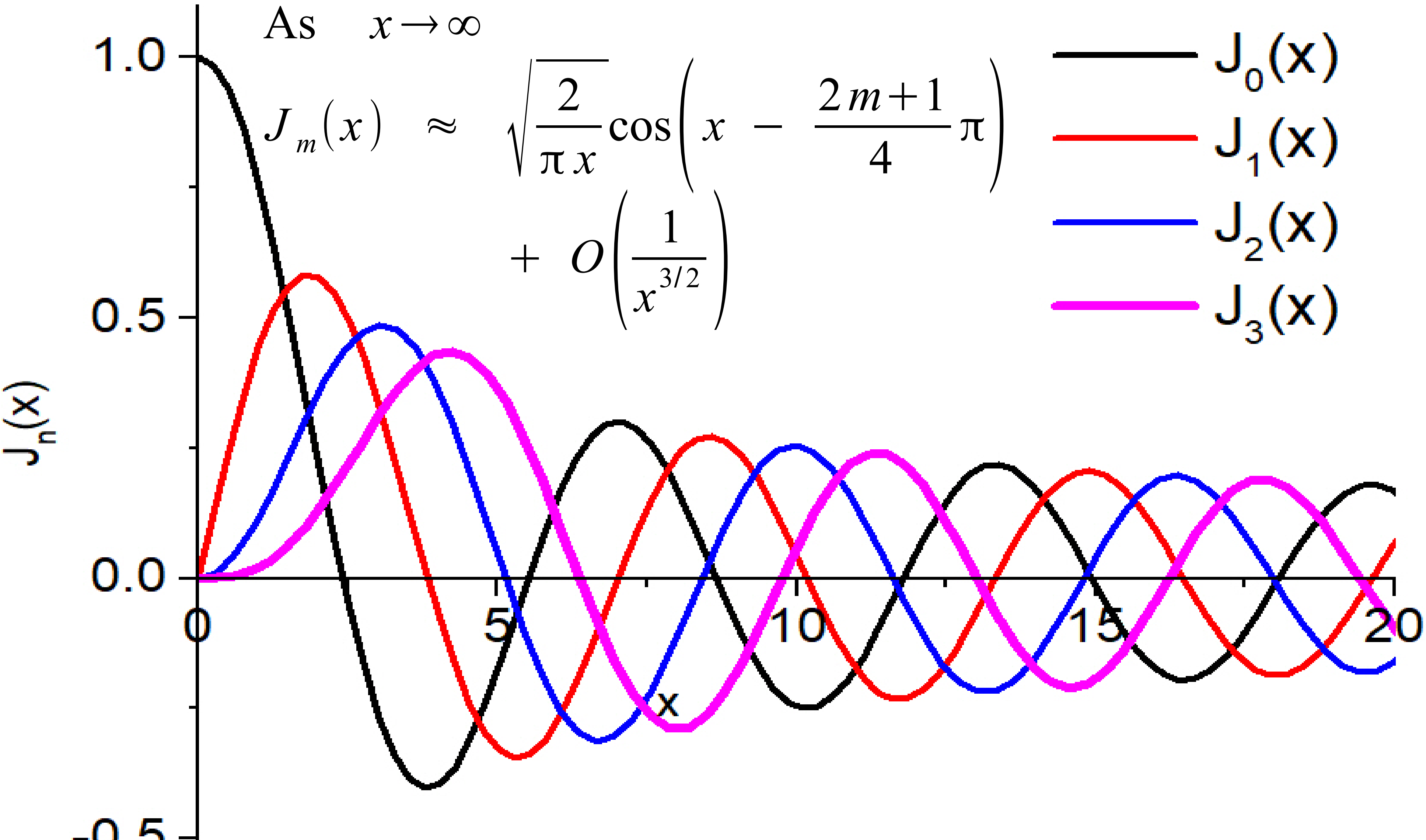
$$\boxed{J_m(x)=\left(\frac{x}{2}\right)^m\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(j+m+1)}\left(\frac{x}{2}\right)^{2j}}$$

How do we know this converges for any x ?

How do Bessel functions look ?

As $x \to \infty$

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2m+1}{4}\pi \right) + O\left(\frac{1}{x^{3/2}} \right)$$



Each of them have infinite number of roots.
No two roots ever coincide except at x=0. (!)

The zeros of Bessel functions

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

The second independent solution

$J_m(x)$ and $J_{-m}(x)$ are linearly independent EXCEPT if m is an integer

$$\begin{aligned}
J_{-m}(x) &= \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j} \\
&= \left(\frac{x}{2}\right)^{-m} \sum_{j=m}^{\infty} \frac{(-1)^j}{j!\,(j-m)!} \left(\frac{x}{2}\right)^{2j} \\
&= \left(\frac{x}{2}\right)^{-m} \sum_{j'=j-m=0}^{\infty} \frac{(-1)^{(j'+m)}}{(j'+m)!\,j'!} \left(\frac{x}{2}\right)^{2(j'+m)} \\
&= (-1)^m \left(\frac{x}{2}\right)^{m} \sum_{j'=0}^{\infty} \frac{(-1)^{j'}}{(j'+m)!\,j'!} \left(\frac{x}{2}\right)^{2j'} \\
&= (-1)^m J_m(x)
\end{aligned}$$

Factorial of negative integer is infinite !

The independent solution for $m=0$ and integers

The second independent solution (Neumann)

$$\pi N_m(x) = \lim_{\nu \to m}\left[\frac{\partial J_\nu}{\partial \nu} - (-1)^\nu \frac{\partial J_{-\nu}}{\partial \nu}\right]$$

See the ref material for more details

$$\pi N_0(x) = 2 J_0(x) \ln\left(\frac{\gamma x}{2}\right) - 2\sum_{j=1}^{\infty}\left[\frac{(-1)^j}{(j!)^2}\left(\frac{x^2}{4}\right)^j \sum_{l=1}^{j}\left(\frac{1}{l}\right)\right]$$

$$\gamma = \text{Euler's constant} = \lim_{n \to \infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln n\right]$$

The derivative with respect to the order of the function may look unusual but it is possible because the order of the Bessel function is defined for any number.

The derivative requires the use of digamma functions

The Neumann solutions are singular at zero. The singularity is logarithmic for m=0, power law like for m=1,2,3...

Factorial of negative integers is infinity (∞)

Define $f(n) \equiv \int_0^\infty x^n e^{-x} dx$

Generalisation of factorial

$$\int_0^\infty x^n e^{-x} dx = \frac{x^{n+1}}{n+1} e^{-x} \Big|_0^\infty + \frac{1}{n+1} \int_0^\infty x^{n+1} e^{-x} dx$$

= 0 if n > -1

$(n+1) f(n) = f(n+1)$ ...Exactly like a factorial.

$$f(0) = \int_0^\infty x^0 e^{-x} dx = 1 \ : (\text{recall } \ 0! = 1)$$

$$f(n) = \int_0^\infty x^n e^{-x} dx \quad \text{converges only if } n > -1$$

$$\frac{1}{f(n)} \to 0 \text{ if } \ n \le -1 \quad \text{is} \quad \frac{1}{n!} \ \text{ for } \ n = 0, 1, 2, 3 \ldots$$

The orthogonality of Bessel functions

The Sturm-Liouville differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dF}{dx}\right] + [q(x) + \mu r(x)]F = 0$$

eigenvalue

weighting function

eigenfunction

$$\int_{x_1}^{x_2} r(x) F_{\mu 1}(x) F_{\mu 2}(x)\, dx = 0 \qquad (\mu_1 \neq \mu_2)$$

orthogonality

The interval (x_1, x_2) has the boundary condition

$$\begin{aligned} a F(x_1) + b F'(x_1) &= 0 \\ c F(x_1) + d F'(x_2) &= 0 \end{aligned} \qquad \begin{cases} a, b, c, d \\ \text{are real constants} \end{cases}$$

sin, cos, Legendre are simple, Bessel needs a bit more work

Sturm Liouville $\rightarrow$ Bessel

$$\frac{d}{dx}\left[p(x)\frac{dF}{dx}\right] + \left[q(x) + \mu\, r(x)\right]F = 0$$

$$\frac{d}{d\rho}\left[\rho\frac{dF}{d\rho}\right] + \left[-\frac{m^2}{\rho} + k^2\rho\right]F = 0$$

$$\text{Orthogonality} \rightarrow \int_{\rho_1}^{\rho_2} \rho\, F(k_1\rho)\, F(k_2\rho)\, d\rho = 0$$

The eigenvalue comes from k NOT from m.
So, solutions corresponding to different k will be orthogonal.
For different 'm', we will get a different set of functions.

The orthogonality of Bessel functions

Recall the substituion we made..... $x = k\rho$

At $\rho = a$

$$J_m\left(\alpha\frac{\rho}{a}\right) = J_m\left(\beta\frac{\rho}{a}\right) = 0 \text{ if } \alpha, \beta \text{ are zeros of } J_m$$

The choice $k = \dfrac{\alpha}{a}$ leads to

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_m\left(\frac{\alpha}{a}\rho\right)\right] + \left(\frac{\alpha^2}{a^2} - \frac{m^2}{\rho^2}\right)J_m\left(\frac{\alpha}{a}\rho\right) = 0$$

Multiply both sides by $\rho J_m\left(\dfrac{\beta}{a}\rho\right)$: integrate over $(0, a)$

Then start with $k = \dfrac{\beta}{a}$ and multiply both sides by $\rho J_m\left(\dfrac{\alpha}{a}\rho\right)$

Then subtract the two results...

The orthogonality of Bessel functions

The orthogonality

$$(\alpha^2-\beta^2)\int_0^a \rho\, J_m\left(\frac{\alpha}{a}\rho\right) J_m\left(\frac{\beta}{a}\rho\right) d\rho = 0$$

The Normalisation

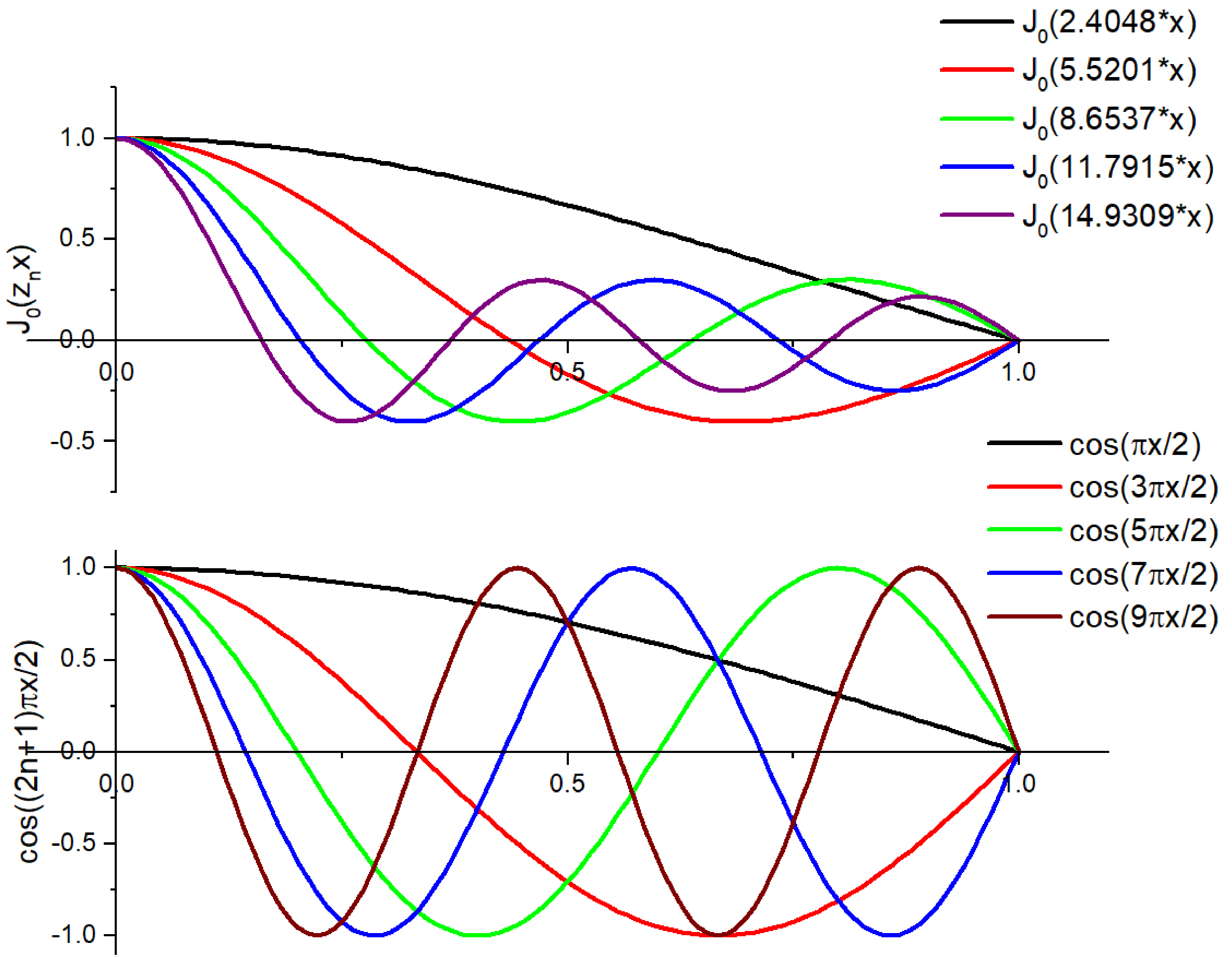
$$\int_0^a \rho\, J_m\left(\frac{\alpha}{a}\rho\right) J_m\left(\frac{\alpha}{a}\rho\right) d\rho = \frac{a^2}{2}\left[J_{m+1}(\alpha)\right]^2$$

Expanding an arbitrary function

$$f(\rho)=\sum_{n=1}^{\infty} A_n J_m\left(\frac{\alpha_n}{a}\rho\right) \qquad \left(\alpha_n \text{is the } n^{th}\ zero\ \text{ of } J_m\right)$$

$$A_n=\frac{2}{a^2\left[J_{m+1}(\alpha_n)\right]^2}\int_0^a \rho\, f(\rho) J_m\left(\frac{\alpha_n}{a}\rho\right) d\rho$$

Geometrically they have similarity to sin/cos



Finally : summary of the solution in (ρ, θ, z)

Assuming $V(\rho, \theta, z)$ is finite at $\rho = 0$ and is zero at $\rho = a$

$$V(\rho, \theta, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \begin{Bmatrix} \sinh k_{mn} z \\ \cosh k_{mn} z \end{Bmatrix} \times$$

$$\left(A_{mn} \cos m\theta + B_{mn} \sin m\theta \right)$$

where $k_{mn} = \dfrac{x_{mn}}{a}$ $\left(x_{mn} \; : \; n^{th} \text{ zero of } \; J_m(x) \right)$

Depending on how the boundary conditions have been provided, one may need to re-write the form of the expression, chose exponential, sinh, cosh etc.

Reminder : summary of the solution in (r, θ, ϕ)

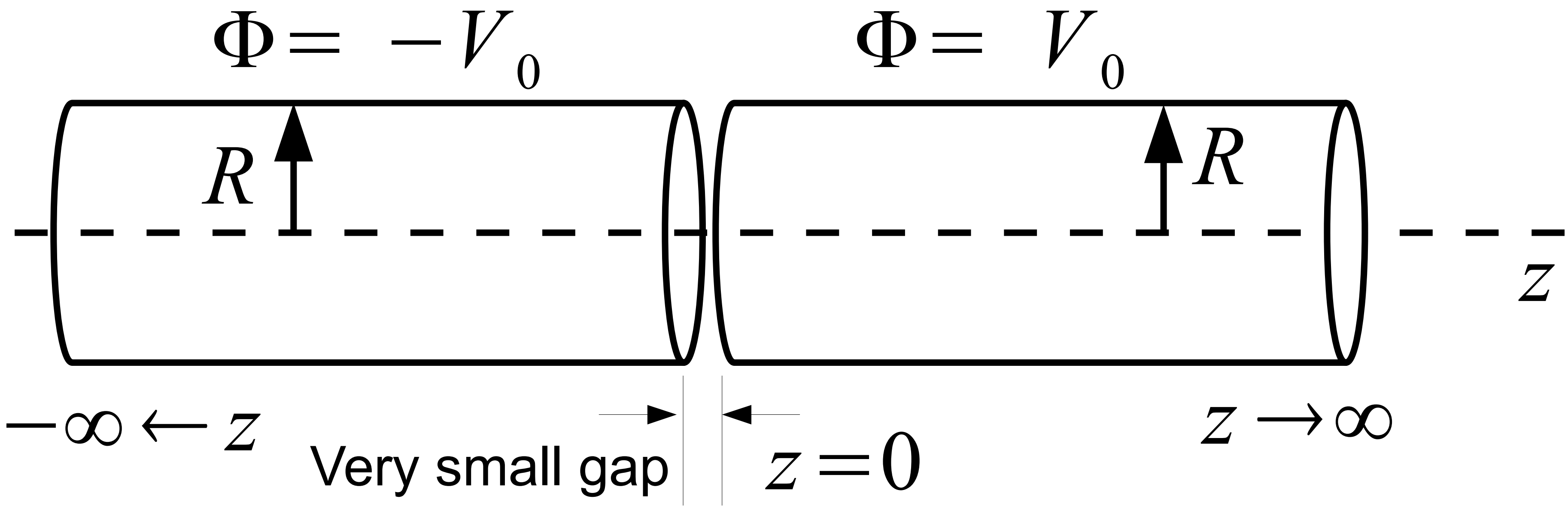
Spherical Polar co-ordinate

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left[A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right] Y_{lm}(\theta, \phi)$$

Generally one finds the coefficients by matching the function on some given spherical surface $r = R$

A tubular lens : solving for the potential



Inside the tube $\quad \Phi = R(\rho)Z(z)$

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad = \quad 0$$

$$\frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} = -\frac{1}{Z}\frac{d^2 Z}{dz^2} \quad = \quad \text{const} = k^2$$

$\Phi(\rho, z \rightarrow \infty)$ is finite $\Rightarrow Z(z) \sim a_k e^{ikz} + b_k e^{-ikz} \Rightarrow k^2 > 0$

$Z(z) \sim e^{\pm|k|z}$ cannot work in this case

A tubular lens : Bessel fn with imaginary arg

The radial solution must be

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} - k^2 = 0$$

compare with

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left(k^2\rho^2 - m^2\right)R = 0$$

$$\Rightarrow \begin{cases} \text{We need} \\ m=0 \\ k \to ik \end{cases}$$

Solution : $\Phi(\rho, z) \sim \sum_k \left(a_k e^{ikz} + b_k e^{-ikz}\right) J_0(ik\rho)$

$\Phi(0, z)$ is finite $\Rightarrow$ no $N_0(ik\rho)$ in solution

There is nothing to force discrete k

$$\Rightarrow \Phi(\rho, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) J_0(ik\rho) e^{ikz}\, dk \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\rho, k) e^{ikz}\, dk$$

A tubular lens : Bessel fn with imaginary arg

Since $\Phi(R,z)$ is known, we can invert the Fourier transform :

$$F(R,k) = \int_{-\infty}^{0} (-V_0) e^{-ikz} dz + \int_{0}^{\infty} (V_0) e^{-ikz} dz$$

$$= \frac{2V_0}{ik} \int_0^\infty \sin u \, du \quad \text{where : } u \equiv kz$$

$$= \lim_{\alpha \to 0} \frac{2V_0}{ik} \int_0^\infty e^{-\alpha u} \sin u \, du = \frac{2V_0}{ik}$$

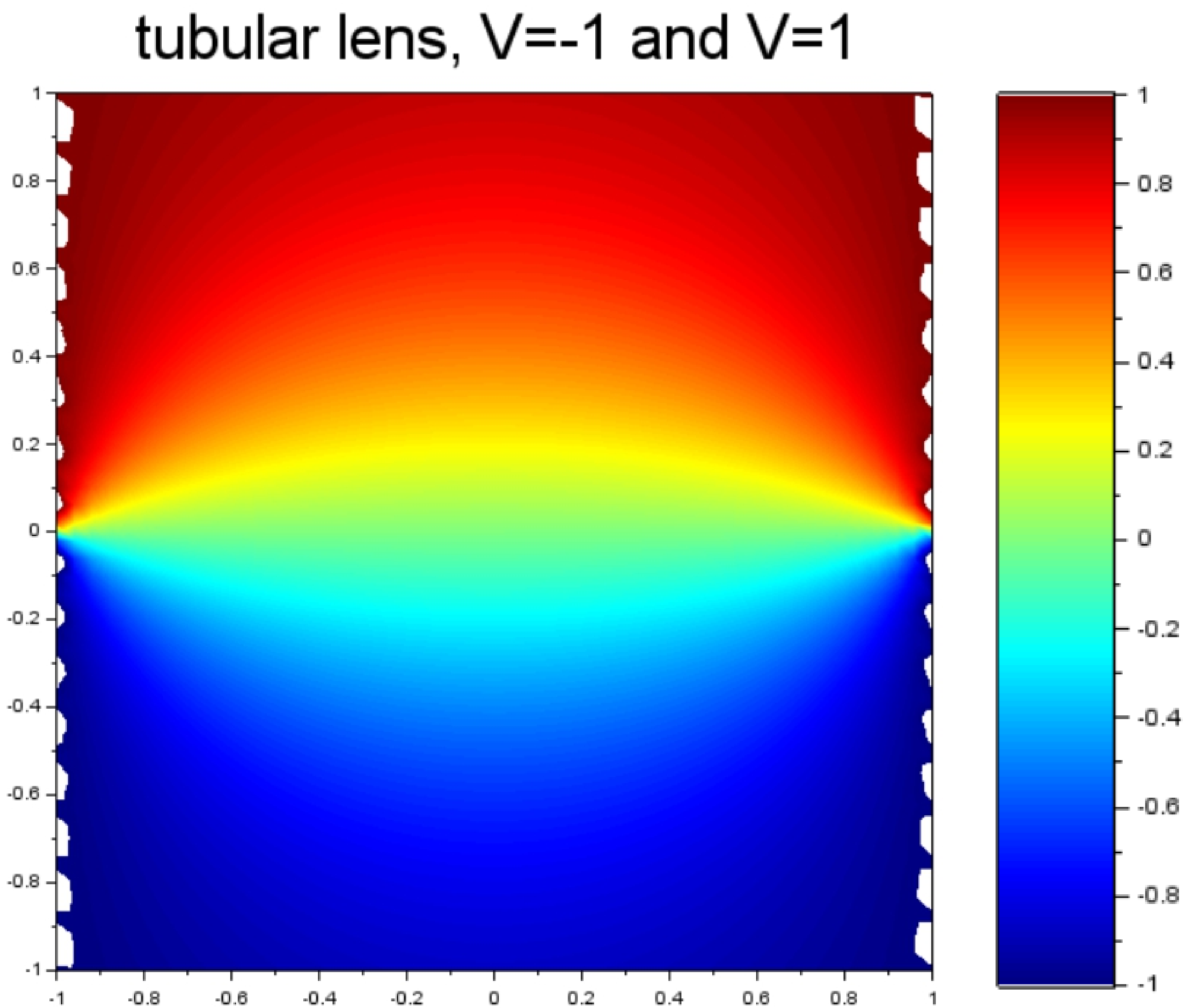
$$\Rightarrow \Phi(\rho, z) = \frac{V_0}{\pi} \int_{-\infty}^{\infty} \frac{J_0(ik\rho)}{ik \, J_0(ikR)} e^{ikz} \, dk$$

Note : $J_0(ik\rho) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!^2} \left(\frac{ik\rho}{2} \right)^{2j} = \sum_{j=0}^{\infty} \frac{1}{j!^2} \left(\frac{k\rho}{2} \right)^{2j}$ NOT oscillatory

Use a script to generate the potential and plot...

```
//Scilab script
function y=pot(rho,z),
y=(0.5/atan(1))*integrate('sin(k*z)*besseli(0,k*rho)/
    (besseli(0,k)*k)','k',0.001,50),
endfunction
clf()
rhorho = linspace(-1,1,50);
zz = linspace(-1,1,50);
set(gcf(),"color_map",jetcolormap(128))
drawlater();
zminmax = [-1 1]; colors=[0 255];
colorbar(zminmax(1),zminmax(2),colors)
Sfgrayplot(rhorho, zz, pot, strf="041",
zminmax=zminmax, colout=[0 0], colminmax=colors)
xtitle("tubular lens, V=-1 and V=1")
drawnow();
show_window()
```

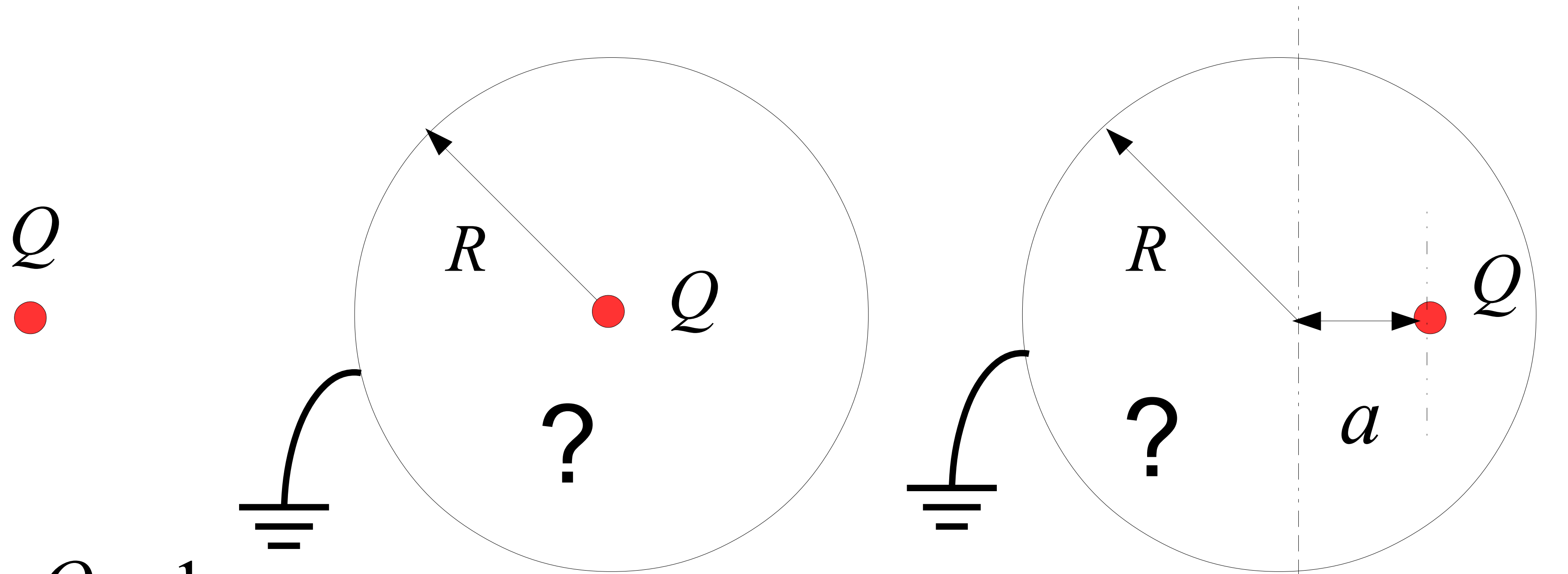
A tubular lens : plot of $\Phi(\rho, z)$



Notice the "lens" shaped curves of the electrostatic potential

Green's function in electrostatics

Charge distribution and boundary condition



$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)$$

The center of the sphere is taken as the origin.
The xy plane only considered

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+y^2}} - \frac{R}{a}\frac{1}{\sqrt{\left(x-\frac{R}{a}\right)^2+y^2}}\right)$$

Boundary condition changes the form of the solution in non-trivial ways.

What does a Green's function do ?

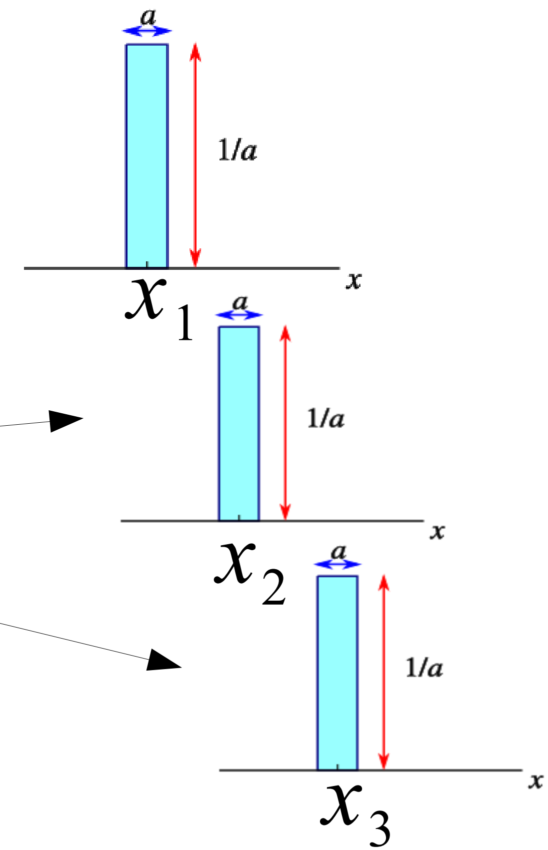
$$[\text{Operator}]\,G(x-x')=-\delta(x-x')$$

Some operator

Some func F1

Some func F2

Some func F3



We assume that superposition works...

Add F1, F2, F3.....to build up LHS

Build up RHS by assembling spikes of different heights.
Like breaking up an integral into rectangles

Can add to G any function that gives RHS = 0

Delta function is a "simple" thing in k-space.

How do we put these together ?

Begin with two arbitrary functions $\psi(r), \phi(r)$

$$\left.\begin{aligned}\nabla.\left(\phi\nabla\psi\right) &= \phi\nabla^2\psi + \nabla\psi.\nabla\phi\\ \nabla.\left(\psi\nabla\phi\right) &= \psi\nabla^2\phi + \nabla\phi.\nabla\psi\end{aligned}\right\}$$

$$\int_{vol}\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)d\tau=\oint_{surf}\left(\phi\nabla\psi-\psi\nabla\phi\right).d\vec{S}$$

Now make the choice $\begin{cases}\psi=G & \text{where} \quad \nabla^2 G(r-r')=-\delta(r-r')\\ \phi=\Phi & \text{where} \quad \nabla^2\Phi=-\dfrac{\rho}{\epsilon_0}\end{cases}$

$$\int_{vol}\left[\Phi\left(-\delta\left(r-r'\right)\right)-G\left(-\frac{\rho}{\epsilon_0}\right)\right]d\tau=\oint_{surf}\left[\Phi\frac{\partial G}{\partial n}-G\frac{\partial\Phi}{\partial n}\right]dS$$

Formal solution in terms of $G(r-r')$

Make a choice $G=0$ on the surface S (Dirichlet)

$$\int_{vol}\left(\Phi\left[-\delta(r-r')\right)-G\left(-\frac{\rho}{\epsilon_0}\right)\right]d\tau=\oint_{surf}\left(\Phi\frac{\partial G}{\partial n}-G\frac{\partial\Phi}{\partial n}\right)dS$$

interchange the role of r and r'

$$\boxed{\Phi(r)=\frac{1}{\epsilon_0}\int_{vol}G(r-r')\rho(r')\,d\tau' - \oint_{surf}\Phi(r')\frac{\partial G}{\partial n}dS'}$$

The formal solution for potential when the charge distribution is given and the potential is specified on the surface S.

But we need to start solving for G in various geometries.

The form of G depends crucially on the boundary conditions!

Interpretation of the terms

$$\Phi(r)=\frac{1}{\epsilon_0}\int_{vol} G(r-r')\rho(r')\,d\tau' - \oint_{surf}\Phi(r')\frac{\partial G}{\partial n}\,dS'$$

The first term gives the contribution of the volume charge.

But imposing a boundary condition (potential) on S requires a (surface) charge distribution to be "pasted" on S. The second term results from that.

If there is no "volume charge", then the potential is entirely determined by the "surface" term. It can be calculated if we know the function G.

Any other possibility ? (von Nuemann...)

Can we make $\dfrac{\partial G}{\partial n}=0$ on the surface S ? !!NO !!

$$\int_{vol}\left(\Phi\left[-\delta(r-r')\right)-G\left(-\frac{\rho}{\epsilon_0}\right)\right]d\tau=\oint_{surf}\left(\Phi\frac{\partial G}{\partial n}-G\frac{\partial\Phi}{\partial n}\right)dS$$

$$\int_{vol}\nabla.(\nabla G)\,d\tau=\boxed{\int_{vol}\nabla^2 G\,d\tau}=\oint_{surf}(\nabla G).d\vec{S}=\boxed{\oint_{surf}\frac{\partial G}{\partial n}dS}$$

Here S is the area of the bounding surface.

$$\boxed{\frac{\partial G}{\partial n}=-\frac{1}{S}\quad:\quad\text{simplest choice}}$$

This choice is used in heat flow related problems. However "mixed boundary value" problems do occur in electostatics. An example is an aperture in a metallic sheet.

$G(\vec{r}-\vec{r}\,')$ for a plane

PROBLEM : The potential is given everywhere on a plane. It is not necessarily constant. How to solve for the potential everywhere?

$$\nabla^2 G = -\delta(x-x')\delta(y-y')\delta(z-z') \;:\; G=0 \text{ if } z=0$$

Dimension of G (for Laplacian) is $[L]$ in 1D, dimensionless in 2D, $[L]^{-1}$ in 3D. Why?

The simplest image charge problem in disguise !
Point charge above a "grounded" conducting plane.

$$G(\vec{r}-\vec{r}\,') = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right)$$

$G(\vec{r}-\vec{r}\,')$ for a plane

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z'}\bigg|_{z'=0} = -\frac{1}{2\pi}\frac{z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

Why is the direction of $\hat{n}$ along $-z'$?

$$\Phi(\vec{r}) = \cancel{\frac{1}{\epsilon_0}\int_{vol} G(r-r')\rho(r')\,d\tau'} - \oint_{surf} \Phi(r')\frac{\partial G}{\partial n}\,dS'$$

$$\Phi(\vec{r}) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\,dy'\,\frac{\Phi(x',y')}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

The potential must be specified everywhere...no holes or slits! The divergence theorem that we used as our starting point, holds only if the surface is closed......

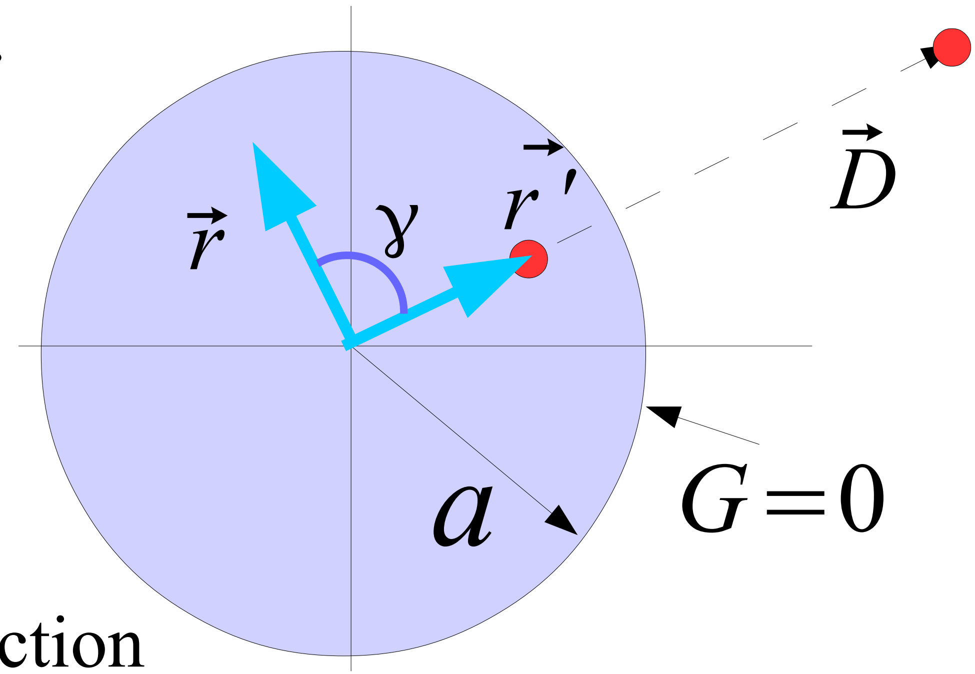
$G(\vec{r}-\vec{r}\,')$ for a sphere

An image charge problem, really...

$$Dr' = a^2 \qquad Q' = -\frac{a}{r'}Q$$

$$\nabla^2 G = \delta(\vec{r}-\vec{r}\,')$$

$$G = \frac{1}{4\pi}\left(\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a/r'}{|\vec{r}-\vec{D}|}\right)$$



since $\vec{r}\,'$ and $\vec{D}$ are in same direction

$$G = \frac{1}{4\pi}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{(rr'/a^2)+a^2-2rr'\cos\gamma}}\right)$$

Normal derivative $$\frac{\partial G}{\partial n} = \begin{cases} \left.\dfrac{\partial G}{\partial r'}\right|_{r'=a} & \text{for} \quad r<a \\ -\left.\dfrac{\partial G}{\partial r'}\right|_{r'=a} & \text{for} \quad r>a \end{cases}$$

r and r' are interchange-able. Why?

$G(\vec{r}-\vec{r}\,')$ for a sphere and spherical harmonics

$$\frac{\partial G}{\partial n} = \begin{cases} \dfrac{1}{4\pi}\dfrac{a - r^2/a}{\left(r^2+a^2-2\,a\,r\cos\gamma\right)^{3/2}} & (r>a) \\ \dfrac{1}{4\pi}\dfrac{r^2/a - a}{\left(r^2+a^2-2\,a\,r\cos\gamma\right)^{3/2}} & (r<a) \end{cases}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

Expressed in spherical harmonics for many calculations.....

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{r^l}{r'^{\,l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') & (r<r') \\ \displaystyle\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{r'^{\,l}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') & (r>r') \end{cases}$$

$G(\vec{r}-\vec{r}\,')$ for a sphere and spherical harmonics

$$G=\frac{1}{4\pi}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}-\frac{a/r'}{|\vec{r}-\vec{D}|}\right) \quad \text{where} \quad D=\frac{a^2}{r'}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\begin{cases}\displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r^l}{r'^{\,l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') & (r<r') \\ \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r'^{\,l}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') & (r>r')\end{cases}$$

$$\frac{a/r'}{|\vec{r}-\vec{D}|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{1}{a}\left(\frac{rr'}{a^2}\right)^l Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')$$

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=a}=\frac{1}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')$$

$G(\vec{r}-\vec{r}\,')$ for a sphere and spherical harmonics

$$\Phi(\vec{r}) = \cancel{\frac{1}{\epsilon_0}\int_{vol} G(r-r')\rho(r')\,d\tau'} \;-\; \oint_{surf} \Phi(r')\frac{\partial G}{\partial n}\,dS'$$

for $r<a$

$$\Phi(\vec{r}) = \sum_{l,m}\left(\frac{r}{a}\right)^{l} Y_{lm}(\theta,\phi)\oint_{surf} d\Omega' Y_{lm}(\theta',\phi')\Phi_s(\theta',\phi')$$

for $r>a$

$$\Phi(\vec{r}) = \sum_{l,m}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\phi)\oint_{surf} d\Omega' Y_{lm}(\theta',\phi')\Phi_s(\theta',\phi')$$

Why is the form different for "interior" and "exterior" points ?

"Multipole form" is useful in telling us the dominant nature of the variation of the potential.

Eignefunction expansion of a δ function

You would have noticed that the functions appearing in the Green's functions are the same functions frequently seen in eignefunction problems. What is the connection ?

Basic fact: We know that any function can be expanded using the "complete" and "orthonormal" set of eignefuctions → So we should be able to expand a delta-fn also in a similar way.

Where does this lead to?

Consider an operator eigenfunction : $L u_n(x) = \lambda_n u_n(x)$

$$\sum_n A_n u_n(x) = f(x) = \delta(x - x')$$

Solve for A_n

Eignefunction expansion of a δ function

$$\int_a^b \sum_n A_n u_n(x) u_m(x)\, dx = \int_a^b \delta(x-x') u_m(x)\, dx$$

$$\sum_n A_n \int_a^b u_n(x) u_m(x)\, dx = u_m(x')$$

Correct normalisation assumed

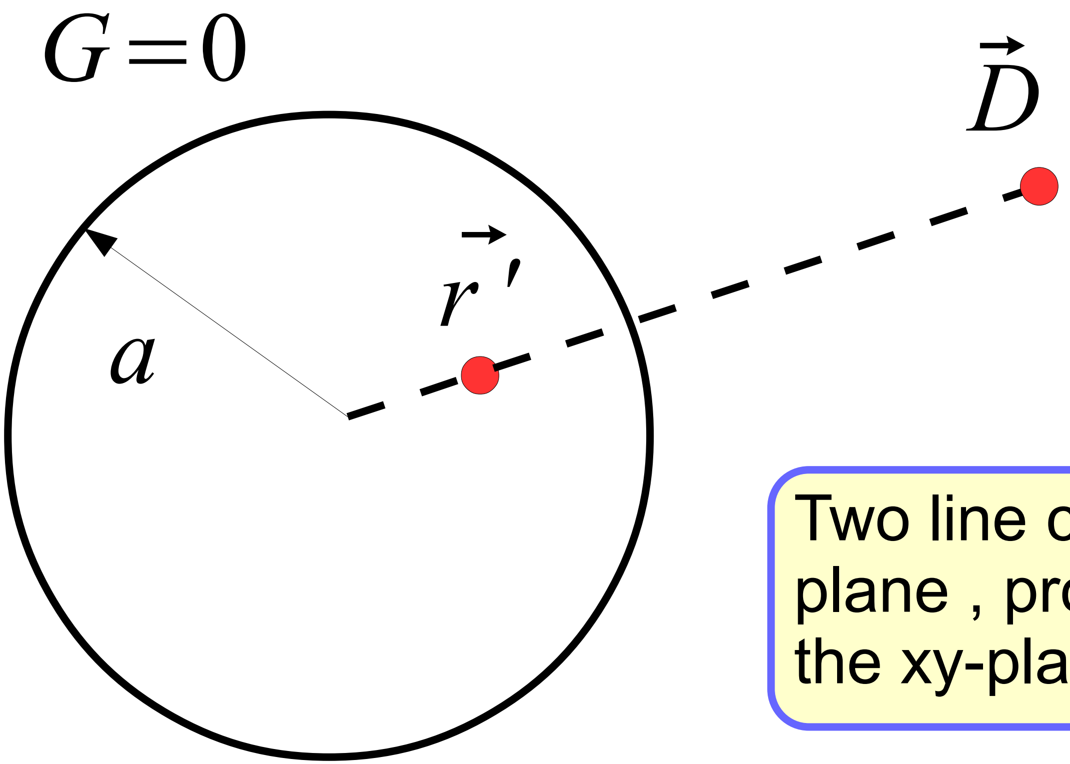
$$\sum_n A_n \delta_{mn} = u_m(x')$$

$$\delta(x-x') = \sum_n u_n(x) u_n(x')$$

So the RHS of a Green's function can be expanded in eigenfuctions for each delta function. The LHS can also be written in terms of eigenfunctions. The solution is guaranteed but not the most handy expression in many cases.

$G(\vec{r}-\vec{r}\,')$ for a long cylinder

The image charge trick works for a "long"/infinite cylinder as a boundary. It does NOT work for a finite sized cylinder .



Cross section of the (hollow) cylinder.

The red dots are "line charges" extending normal to the plane of the paper..

Two line charges perpendicular to the xy plane , produce circular equipotentials in the xy-plane. Utilise this....

Consider the function

$$g=\ln\left(\left|\vec{r}-\vec{r}\,'\right|\right)-\ln\left(\left|\vec{r}-\vec{D}\right|\right)$$

ln is a solution of Laplace eqn in 2D polar

$G(\vec{r}-\vec{r}\,')$ for a long cylinder

$$g = \ln\left(\left|\vec{r}-\vec{r}\,'\right|\right)-\ln\left(\left|\vec{r}-\vec{D}\right|\right)$$

$$= \frac{1}{2}\ln\left(\frac{a^2+r'^2-2\,a\,r'\cos\gamma}{a^2+D^2-2\,a\,D\cos\gamma}\right) \quad \text{if } \; r=a$$

$$= \frac{1}{2}\ln\left(\frac{r'^2}{a^2}\times\frac{\dfrac{a^2}{r'^2}+1-2\dfrac{a}{r'}\cos\gamma}{1+\dfrac{D^2}{a^2}-2\dfrac{D}{a}\cos\gamma}\right) \qquad \text{choose} \quad \frac{D}{a}=\frac{a}{r'}$$

$$= \ln\left(\frac{r'}{a}\right) \qquad \text{subtract this from } g \text{ to get } G(\rho=a)=0$$

Introduce the usual cylindrical polar (ρ,θ) variables

$$G = -\frac{1}{2\pi}\ln\left(\frac{\sqrt{\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\theta-\theta')}}{\sqrt{(\rho\,\rho'/a)^2+a^2-2\,\rho\,\rho'\cos(\theta-\theta')}}\right)$$

$G(\vec{r}-\vec{r}\,')$ for a long cylinder

$$\frac{\partial G}{\partial n} = -\left.\frac{\partial G}{\partial \rho\,'}\right|_{\rho\,'=a^+} = \frac{1}{2\pi}\frac{a-\rho^2/a}{\rho^2+a^2-2a\rho\cos(\theta-\theta\,')} \qquad (\rho>a)$$

$$\frac{\partial G}{\partial n} = \left.\frac{\partial G}{\partial \rho\,'}\right|_{\rho\,'=a} = \frac{1}{2\pi}\frac{\rho^2/a-a}{\rho^2+a^2-2a\rho\cos(\theta-\theta\,')} \qquad (\rho<a)$$

$$\begin{aligned}\Phi(\rho,\theta) &= -\oint \Phi(\theta\,')\frac{\partial G}{\partial n}\,a\,d\theta\,' \\ &= \frac{1}{2\pi}\oint_0^{2\pi}\Phi(\theta\,')\frac{a^2-\rho^2}{\rho^2+a^2-2a\rho\cos(\theta-\theta\,')}\,d\theta\,'\end{aligned}$$

This is exactly the Poisson integral formula, as expected

What should a δ function look like in (ρ, θ, z)?

$$\int_{vol} \delta(\vec{r}-\vec{r}\,')\, d\tau = 1 \qquad \text{must hold}$$

$$\int_{vol} \frac{\delta(\rho-\rho')}{\rho} \delta(\theta-\theta')\delta(z-z')\, d\rho\, \rho\, d\theta\, dz$$

In general, for $u_{1,} u_{2,} u_3$

$$d\tau = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(u_1-u_1')}{h_1}\frac{\delta(u_2-u_2')}{h_2}\frac{\delta(u_3-u_3')}{h_3}$$

It is possible to integrate out angular co-ordiantes if there is no angle depndence of the functions that are being dealt with. For example

$$\delta(\vec{r}-\vec{r}\,') \rightarrow \frac{1}{4\pi r^2}\delta(r-r') \quad \text{in } (r, \theta, \phi) \text{ with only } r \text{ dependenc}$$

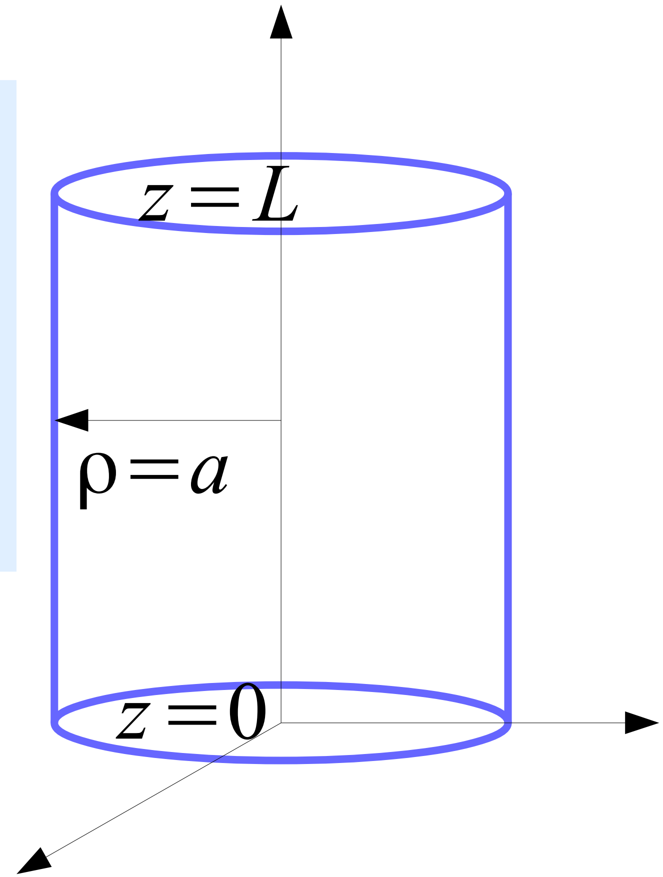
$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

We need to solve

$$\nabla^2 G = \left(\frac{\partial}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial}{\partial\theta^2}+\frac{\partial}{\partial z^2}\right)G$$

$$= -\frac{\delta(\rho-\rho')}{\rho}\delta(\theta-\theta')\delta(z-z')$$

$G = 0$ on all surfaces of the cylinder



$G=R(\rho,\rho')\Phi(\theta,\theta')Z(z,z')$ works.....

But we are NOT going to get decoupled equations like

$$\left(\quad\right)R(\rho,\rho') = \frac{\delta(\rho-\rho')}{\rho}$$

$$\left(\quad\right)\Phi(\theta,\theta') = \delta(\theta-\theta')$$

$$\left(\quad\right)Z(z,z') = \delta(z-z')$$

The RHS will be zero for decoupling the equation, but each equation will be solved twice...once each for two sides of the delta-fn

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

$$\frac{d^2}{dz^2}Z(z,z')-k^2Z(z,z')=0 \qquad \text{Solution}$$

$$Z_k=\begin{cases} A_k(z')\sinh kz \quad + \quad \cancel{B_k(z')\cosh kz} & (0<z<z'<L) \\ C_k(z')\sinh kz \quad + \quad D_k(z')\cosh kz & (0<z'<z<L) \end{cases}$$

$$Z_k(0,z') \quad = \quad 0 \quad \Rightarrow \quad B_k(z')=0$$

$$Z_k(L,z') \quad = \quad 0 \quad \Rightarrow \quad C_k(z')=-\frac{\cosh kL}{\sinh kL}D_k(z')$$

G is continous at z' $\Rightarrow$ $A_k(z')=D_k(z')\dfrac{\sinh k(L-z')}{\sinh kz'\sinh kL}$

So, except one all coefficients have been solved for.
What condition should determine that?

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

$$Z_k = \begin{cases} D_k(z')\dfrac{\sinh k(L-z')}{\sinh kz'\sinh kL}\sinh kz & (0<z<z'<L) \\ D_k(z')\dfrac{\sinh k(L-z)}{\sinh kL} & (0<z'<z<L) \end{cases}$$

How do we use the symmtery Z(z,z') = Z(z',z) ?
If we interchange the values of z and z', then the solution for z < z' must produce the solution for z > z'

$$D_k(z)\frac{\cancel{\sinh k(L-z)}}{\sinh kz\,\cancel{\sinh kL}}\sinh kz' = D_k(z')\frac{\cancel{\sinh k(L-z)}}{\cancel{\sinh kL}}$$

$$Z_k = \begin{cases} \dfrac{\sinh k(L-z')}{\sinh kL}\sinh kz & (0<z<z'<L) \\ \dfrac{\sinh k(L-z)}{\sinh kL}\sinh kz' & (0<z'<z<L) \end{cases}$$

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

Repeat exactly the same process for the $\Phi(\theta,\theta')$ part

$$\frac{d^2}{d\phi^2}\Phi(\theta,\theta')-m^2\Phi(\theta,\theta')=0 \text{ gives}$$

$$\Phi=\cos m(\theta-\theta') \qquad (m=0,\pm1,\pm2,.\ ....)$$

The radial part

$$\rho^2\frac{d^2}{d\rho^2}R+\rho\frac{d}{d\rho}R+\left(k^2\rho^2-m^2\right)R=0 \qquad \text{is solved by}$$

$$R(\rho,\rho')=\begin{cases} A_m(\rho')J_m(k\rho) & 0<\rho<\rho'<a \\ C_m(\rho')J_m(k\rho) & 0<\rho'<\rho<a \end{cases}$$

$N_m(\rho=0)$ diverges. So not part of the solution .

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

$$\left.\begin{array}{l}\text{Continuity at } \rho=\rho' \\ \text{Symmetry } \rho\Leftrightarrow\rho' \\ R(\rho=a,\rho')=0\end{array}\right\}\Rightarrow \quad \begin{array}{rcl} R(\rho,\rho') &=& J_m(k\rho')J_m(k\rho) \\ k &=& \dfrac{x_{mn}}{a} \; : \; n^{th} \text{ zero of } J_m(x)\end{array}$$

The full solution is obtained by combining

$$G(\vec{r},\vec{r}\,')=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}A_{mn}J_m(k_{mn}\rho')J_m(k_{mn}\rho)Z(z,z')\cos m(\theta-\theta')$$

The coefficients A_{mn} will ensure the δ functions on RHS

For $\delta(\theta-\theta')$: multiply both sides by $\cos p\theta$ and integrate

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

For $\delta(\theta-\theta')$: allow the θ derivative to work
then multiply both sides by $\cos p(\theta-\theta')$ and integrate

$$\sum_{mn} A_{mn} \int_0^{2\pi} \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}(-m^2) + \frac{\partial^2}{\partial z^2} \right) R\,\Phi\,Z \cos p(\theta-\theta')\,d\theta =$$

$$-\int_0^{2\pi} \frac{\delta(\rho-\rho')}{\rho}\delta(\theta-\theta')\delta(z-z')\cos p(\theta-\theta')\,d\theta$$

$$\sum_n \pi A_{pn} \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + \frac{\partial^2}{\partial z^2} \right) R(\rho,\rho')Z(z,z') =$$

$$\frac{\delta(\rho-\rho')}{\rho}\delta(z-z')$$

For $\delta(z-z')$: integrate both sides between $z \pm \epsilon$

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

$$\sum_n \pi A_{pn} \int_{z'-\epsilon}^{z'+\epsilon} \left(\frac{\partial}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{p^2}{\rho^2} + \frac{\partial}{\partial z^2} \right) RZ\, dz \quad =$$

$$\int_{z-\epsilon}^{z+\epsilon} dz \left[-\frac{\delta(\rho-\rho')}{\rho} \delta(z-z') \right]$$

These terms cannot contribute to the integral becuase Z(z.z') is continous at z=z'.
So only contribution can come from the z-derivative, becuase Z has a DIFFERENT functional form for z<z' and z>z'

$$\sum_n \pi A_{pn} R \left(\frac{dZ}{dz} \Bigg|_{z'+\epsilon} - \frac{dZ}{dz} \Bigg|_{z'-\epsilon} \right) = -\frac{\delta(\rho-\rho')}{\rho}$$

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

$$Z_k = \begin{cases} \dfrac{\sinh k(L-z')}{\sinh kL}\sinh kz & (0<z<z'<L) \\ \dfrac{\sinh k(L-z)}{\sinh kL}\sinh kz' & (0<z'<z<L) \end{cases}$$

$$\left(\frac{dZ}{dz}\bigg|_{z+\epsilon} - \frac{dZ}{dz}\bigg|_{z-\epsilon}\right) = -k = \left(\frac{x_{pn}}{a}\right)$$

$$\boxed{R(\rho,\rho') = J_m(k\rho')J_m(k\rho)}$$

$$\sum_n A_{pn} R\,\pi\left(-\frac{x_{pn}}{a}\right) = -\frac{\delta(\rho-\rho')}{\rho}$$

$$\int_0^a d\rho\;\;\rho J_p\left(\frac{x_{pq}}{a}\rho\right)\sum_n A_{pn} J_p\left(\frac{x_{pn}}{a}\rho'\right) J_p\left(\frac{x_{pn}}{a}\rho\right)\pi\left(\frac{x_{pn}}{a}\right) =$$

$$\int_0^a d\rho\,\rho J_p\left(\frac{x_{pq}}{a}\rho\right)\frac{\delta(\rho-\rho')}{\rho} = J_p\left(\frac{x_{pq}}{a}\rho'\right)$$

$G(\vec{r}-\vec{r}\,')$ for the interior of finite cylinder

$$\sum_n \left(\frac{\pi x_{pn}}{a}\right) A_{pn} J_p\left(\frac{x_{pn}}{a}\rho'\right) \int_0^a d\rho\;\rho J_p\left(\frac{x_{pq}}{a}\rho\right) J_p\left(\frac{x_{pn}}{a}\rho\right) = J_p\left(\frac{x_{pq}}{a}\rho'\right)$$

Using $\displaystyle\int_0^a d\rho\;\rho J_p\left(\frac{x_{pq}}{a}\rho\right) J_p\left(\frac{x_{pn}}{a}\rho\right) = \frac{a^2}{2} J^2_{p+1}\left(x_{pn}\right)\delta_{nq}$

We get the solution for A_{pn}

$$A_{pn} = \frac{1}{\pi a^2 k_{pn}} \frac{1}{J^2_{p+1}(k_{pn} a)}, \qquad p=0, \quad k_{pn} = \frac{x_{pn}}{a}$$

$$= \frac{2}{\pi a^2 k_{pn}} \frac{1}{J^2_{p+1}(k_{pn} a)} \qquad p \geq 1$$

$$G(\vec{r},\vec{r}\,') = \frac{1}{\pi a} \sum_{p=0}^{\infty} \sum_{n=1}^{\infty} \frac{\left(2-\delta_{p0}\right)}{x_{pn}} \left[\frac{J_p(k_{pn}\rho')J_p(k_{pn}\rho)}{J^2_{p+1}(x_{pn})}\right] Z(z,z')\cos p(\theta-\theta')$$

PART 2:

Energy, momentum and force in electromagnetism

E,D,B,H

Wave propagation, reflection & refraction

Energy conservation

Conservative field $\rightarrow$ KE + PE (scalar potential) conserved.
EM fields are in general not conservative, so what is conserved?

So may be : KE of particles + "something" will be conserved ?

$$\delta W_M = \int_{all\,vol} \rho(\vec{E}+\vec{v}\times\vec{B}).\vec{v}\,\delta t\, d\tau$$

Work done on the charges= Force x displacement (integrated over all vol)

$$\frac{dW_M}{dt} = \int \vec{E}.\vec{j}\, d\tau$$

$Use\ \nabla\times\vec{B} = \mu_0\vec{j}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$ to replace $\vec{j}$

$$\nabla.(\vec{E}\times\vec{B}) = \vec{B}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{B}$$

$$= \frac{1}{\mu_0}\int(\vec{E}.\nabla\times\vec{B})d\tau - \frac{\partial}{\partial t}\int\frac{\epsilon_0 E^2}{2}d\tau$$

$$= -\frac{1}{\mu_0}\int\nabla.(\vec{E}\times\vec{B})d\tau + \frac{1}{\mu_0}\int\vec{B}.(\nabla\times\vec{E})d\tau$$

$$- \frac{\partial}{\partial t}\int\frac{\epsilon_0 E^2}{2}d\tau$$

Energy conservation

$$\frac{dW_M}{dt} = -\frac{1}{\mu_0}\int \nabla.(\vec{E}\times\vec{B})\,d\tau + \frac{1}{\mu_0}\int \vec{B}.(\nabla\times\vec{E})\,d\tau$$

$$- \frac{\partial}{\partial t}\int \frac{\epsilon_0 E^2}{2}\,d\tau$$

$$= -\frac{1}{\mu_0}\int \nabla.(\vec{E}\times\vec{B})\,d\tau - \frac{\partial}{\partial t}\int\left(\frac{\epsilon_0 E^2}{2}+\frac{B^2}{2\mu_0}\right)d\tau$$

$$\therefore \quad \frac{d}{dt}\left[W_M + \int_{vol}\left(\frac{\epsilon_0 E^2}{2}+\frac{B^2}{2\mu_0}\right)d\tau\right] = -\frac{1}{\mu_0}\int_{surf} \nabla.(\vec{E}\times\vec{B})\,d\tau$$

compare with $\nabla.\vec{j}+\frac{\partial\rho}{\partial t} = 0$: OR : $\frac{dQ_{\text{in}}}{dt} = -\int_{surf} \vec{j}.d\vec{a}$

Momentum conservation

We find that the EM field contains energy and we can identify the energy flux/flow/current term as well.

Natural question: Can we do the same for momentum of the particles? This is more invloved, becuase momentum is a vector and forming the continuity equation for a vector would require a "tensor".

Apart from that the reasoning is very similar...

$$\frac{d}{dt}\sum_{all}\vec{p}_i = \vec{F} = \int_{all\ \ vol} \rho\left(\vec{E}+\vec{v}\times\vec{B}\right)d\tau$$

$$= \int\left[\left(\epsilon_0\nabla.\vec{E}\right)\vec{E} + \left(\frac{\nabla\times\vec{B}}{\mu_0}-\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]d\tau$$

Momentum conservation

$$\frac{d}{dt}\sum_{all}\vec{p}_i = \vec{F} = \int_{all\ \ vol} \rho(\vec{E}+\vec{v}\times\vec{B})\,d\tau$$

$$= \int\left[\left(\epsilon_0\nabla.\vec{E}\right)\vec{E} + \left(\frac{\nabla\times\vec{B}}{\mu_0}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right)\right]d\tau$$

Since : $(\nabla\times\vec{B})\times\vec{B} = (\vec{B}.\nabla)\vec{B} - \nabla\frac{B^2}{2}$

And : $\left(\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\nabla\times\vec{E})$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \left[(\vec{E}.\nabla)\vec{E} - \nabla\frac{E^2}{2}\right]$$

Momentum conservation

RHS becomes :

$$\epsilon_0\left[(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]+\frac{1}{\mu_0}\left[(\nabla.\vec{B})\vec{B}+(\vec{B}.\nabla)\vec{B}-\nabla\frac{B^2}{2}\right]-\frac{1}{c^2}\frac{\partial}{\partial t}\frac{(\vec{E}\times\vec{B})}{\mu_0}$$

The integrand is now symmetric in E and B although the initial expression was not. The extra term we have added is div B which is always zero.

S = E x **B** emerges again

$$\frac{d}{dt}\left[\sum_{particles}\vec{p}_i+\frac{1}{c^2}\int\vec{S}\,d\tau\right]=\int\left[\epsilon_0\left\{(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right\}+\frac{1}{\mu_0}\left\{(\nabla.\vec{B})\vec{B}+(\vec{B}.\nabla)\vec{B}-\nabla\frac{B^2}{2}\right\}\right]d\tau$$

Question : Is RHS the divergence of something? Then the form of the continuity equation will emerge again.

But the RHS is already a vector, so it can only be the divergence of tensor (if at all)

Momentum conservation

$$\left[(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]_i = \frac{\partial E_j}{\partial x_j}E_i+E_j\frac{\partial E_i}{\partial x_j}-\frac{1}{2}\frac{\partial E^2}{\partial x_i}$$

$$=\frac{\partial}{\partial x_j}\left(E_iE_j-\delta_{ij}\frac{E^2}{2}\right)$$

Repeated index j is summed over, there is no summation over i

Hence entire RHS integrand is a divergence of the following

$$T_{ij} = \epsilon_0\left(E_iE_j-\delta_{ij}\frac{E^2}{2}\right)+\frac{1}{\mu_0}\left(B_iB_j-\delta_{ij}\frac{B^2}{2}\right)$$

Q: Why would you call it a stress tensor?

$$\frac{d}{dt}\left[\sum_{particles}\vec{p}_i+\frac{1}{c^2}\int\vec{S}\,d\tau\right] = -\int_{vol}\nabla.(-\underline{\underline{T}})\,d\tau = -\int_{surf}(-\underline{\underline{T}}).d\vec{a}$$

compare with $\frac{d}{dt}Q_{\text{inside}} = -\int_{vol}\nabla.\vec{j}\,d\tau = -\int_{surf}\vec{j}.d\vec{a}$

Electrodynamics and materials

In any material there are huge number of charges (nucleii + electrons). A complete description of the electrodynamics of a "material" should take these into account!

This is the exact "microscopic" description. In this description there is only E, B and fundamental constants. There is no D and H. (see Classical Electrodynamics , sec 6.7 – 6.9 : J.D. Jackson)

This is clearly impractical. So we invent some ways of retaining the form of the Maxwell's equations, but introduce some paramters and very few new variables.

It works well for many cases (refractive index for light is a very good example.) It will not work when the atomistic "discreteness" is important (X-ray diffraction)

Electrodynamics and materials

The average description relies on modelling the "charge/magnetisation neutral" background as something that develops a small electric/magnetic dipole moment.

The "bound charge" and "bound currents" that we talk about are essentially these dipoles. For this approach to work it must be easy to separate out what is "bound" and what is free. This Maxwell's equations do not tell us. We have to decide.

Linear dependence of polarizability and magnetisation is not necessary but simplifies the formulation a lot

The definition of $\vec{D}$

$$\nabla . \vec{E} = \frac{\rho_{TOTAL}}{\epsilon_0} \Rightarrow \nabla . \epsilon_0 \vec{E} = \rho_{free} + \rho_{pol}$$

since $\rho_{pol} = -\nabla . \vec{P}$, we can write

$$\nabla . \left[\epsilon_0 \vec{E} + \vec{P}\right] = \rho_{free} \quad \text{OR} \quad \nabla . \vec{D} = \rho_{free}$$

Use the proprotionality of $\vec{P}$ *with* $\vec{E}$:

$\vec{P} = \epsilon_0 \chi \vec{E}$ (This is phenomenological)

$\epsilon_0 (1+\chi) \vec{E} = \epsilon \vec{E} = \vec{D}$ (Linear material)

Quantities like D, ε can only be defined in an average sense.
Makes sense if averaged over a few (~10 -100) lattice units.

!! One cannot talk about D or ε inside an atom!!

The definition of $\vec{H}$

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 \left(\vec{J}_f + \vec{J}_b \right)$$

"Free" current put in by wires, solenoids etc.

"Bound" current due to induced or frozen magnetic dipoles

$$\vec{J}_b = \nabla \times \vec{M} \qquad hence$$

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f$$

$$call \quad \frac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}$$

$$\nabla \times \vec{H} = \vec{J}_f$$

$$\nabla . \vec{H} = ?$$

A proportionality between **M** and **H** is a material property.

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

$$\vec{M} = \chi \vec{H}$$

$$\vec{B} = \mu_0 \left(1 + \chi \right) \vec{H}$$

$$\vec{B} = \mu \vec{H}$$

χ is called susceptibility

μ is called permeability

Maxwell's equation does NOT tell you how to distinguish "free" and "bound" current.

Maxwell's equations with $\vec{E}, \vec{D}, \vec{B}, \vec{H}$

Consider an insulator, so there are no free charges in the material

$$\vec{D} = \epsilon \vec{E} \qquad \nabla . \vec{D} = 0$$

$$\vec{B} = \mu \vec{H} \qquad \nabla . \vec{B} = 0$$

Magnetisation and electric polarisation can simultaneously change. So the "bound" current will result from change in **M** as well as **P**.

$\sigma_b = \vec{P} . \hat{n}$: Then consider $\vec{P} \rightarrow \vec{P} + \delta \vec{P}$

This change causes some amount of charge to flow in/out

$$\delta Q = \delta(\vec{P} . \hat{n}) \delta a \quad \text{hence} \quad \vec{J}_p . \delta \vec{a} = \frac{\delta Q}{\delta t} = \frac{\partial \vec{P}}{\partial t} . \delta \vec{a}$$

Total bound current flow $\qquad \vec{J}_b = \nabla \times \vec{M} + \dfrac{\partial \vec{P}}{\partial t}$

Maxwell's equations with $\vec{E}, \vec{D}, \vec{B}, \vec{H}$

$$\nabla\times\vec{B} = \mu_0\vec{J}_{total} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\left[\mu_0(\vec{H}+\vec{M})\right] = \mu_0\left[\vec{J}_f+\nabla\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right] + \mu_0\frac{\partial}{\partial t}\left[\vec{D}-\vec{P}\right]$$

$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla.\vec{D} = \rho_f \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla.\vec{B} = 0 \qquad \nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

From the Energy point of view

What happens to : $$U = \int_{vol} \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) d\tau$$

We still expect energy to be a quadratic function of the field strength.

Suppose we change E(x,y,z) by a small amount dE(x,y,z) and ask the question : How much work has been done in the process?

But the question really is "How much work has been done on/by the free charges?“ This is because the "free charge" is what the experimenter can control.

Mathematically the "functional derivative" of U w.r.t. E is the answer to the question. We can show that the function D treated as a function of E is the answer. It can be taken as a definition of D too.

Treat $\vec{D}$ as a functional derivative

$$\delta U = \int_{vol} f(\vec{E}).\delta\vec{E}\,d\tau \qquad (\vec{E} \to \vec{E} + \delta\vec{E})$$

$$= \int_{vol} f(\vec{E}).(-\nabla\delta V)\,d\tau \qquad (\vec{E} = -\nabla V)$$

$$= \int_{vol} \left[\delta V\,\nabla.\vec{f} - \nabla.\left(\vec{f}\,\delta V\right)\right] d\tau$$

$$= \int_{vol} \delta V\,\nabla.\vec{f}\,d\tau - \int_{surf} \vec{f}.d\vec{S}$$

Second term → zero if we take the volume large enough such that the fields have all gone to zero.

First term: identify f(E) = D . Then div D should gives the free charge density and the expression gives the increase in energy of the free charges .

Treat $\vec{H}$ as a functional derivative

$$\delta U = \int_{vol} f(\vec{B}).\delta\vec{B}\,d\tau \qquad (\vec{B}\rightarrow\vec{B} + \delta\vec{B})$$

$$= \int_{vol} f(\vec{B}).\left(-\nabla\times\vec{E}\right)\delta t\,d\tau \qquad \left(\nabla\times\vec{E} = -\frac{\delta\vec{B}}{\delta t}\right)$$

$$= \int_{vol} \left[\vec{E}.\nabla\times\vec{f} - \nabla.\left(\vec{f}\times\vec{E}\right)\right]d\tau$$

$$= \int_{vol} \vec{E}.\nabla\times\vec{f}\,d\tau - \int_{surf} \vec{f}\times\vec{E}.d\vec{S}$$

Second term $\rightarrow$ zero if we take the volume large enough such that the fields have all gone to zero.

First term identify f = H . Then curl f should gives the free current density and the expression gives the increase in energy (work done on) of the free current .

Important to remember about $\vec{H}$ and $\vec{B}$

All currents contribute to curl **B**, but only the external current (typically current in wires/coils) contributes to curl **H**. It is tempting to say that **H** is the field that would exist if the magnetic materials were not put in there. This is NOT in general correct.

If the sample is long and cylindrical then it is correct, but for NO other shape. The complete solution, when a sample is placed in an "initially uniform" field is possible for a sphere and a few other shapes.

However, the statement "**H** is the field in a medium" is WRONG !!

In cases where "permanent magnets" are there, it is more complex. In fact in a permanent magnet **H** and **B** may point in opposite directions.

The expression for linear media

$$\delta U = \int_{vol} \left[\vec{D}\cdot\delta\vec{E} + \vec{H}\cdot\delta\vec{B}\right] d\tau$$

Holds for linear or non-linear medium

If $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$

$$U = \int_{vol} \left[\frac{\epsilon}{2}\vec{E}\cdot\vec{E} + \frac{\mu}{2}\vec{H}\cdot\vec{H}\right] d\tau$$

$$= \int_{vol} \frac{1}{2}\left[\vec{D}\cdot\vec{E} + \vec{H}\cdot\vec{B}\right] d\tau$$

$$u = \frac{1}{2}\left[\vec{D}\cdot\vec{E} + \vec{H}\cdot\vec{B}\right]$$

A very commonly used expression. But this works for linear media only.

ϵ and μ are in general symmetric tensors

In general ϵ and μ are tensors (3×3) matrices

$$\Rightarrow u = \frac{1}{2}\left[\vec{E}\,\epsilon\,\vec{E}\,. + \vec{H}\,\mu\,\vec{H}\right]$$

Energy conservation in a case with $\vec{H} = const$

No time dependance

$$\frac{\partial u}{\partial t} + \nabla.(\vec{E}\times\vec{H}) = 0$$

$$\sum_{ij}\frac{\epsilon_{ij}}{2}\left[E_i\frac{\partial E_j}{\partial t} + E_j\frac{\partial E_i}{\partial t}\right] - \vec{E}.\frac{\partial\vec{D}}{\partial t} = 0$$

$$\sum_{ij}\frac{\epsilon_{ij}+\epsilon_{ji}}{2}E_i\frac{\partial E_j}{\partial t} - \sum_{ij}\epsilon_{ij}E_i\frac{\partial E_j}{\partial t} = 0$$

$$\boxed{\Rightarrow \epsilon_{ij} = \epsilon_{ji} \qquad (\text{similar proof for } \mu_{ij} = \mu_{ji})}$$

What about momentum ?

Viewpoint 1: The speed of light has been slowed down by a factor of n (the refractive index) .So momentum will REDUCE by a factor of n, like that of any classical "particle"

Viewpoint 2: The wavelength of light has changed from λ to λ/n. So the wavevector (k) must have become LARGER in magnitude. We know that momentum is proportional to the wavevector, hence momentum should INCREASE ?

The momentum of light in a medium

In most media $\epsilon_0 \rightarrow \epsilon$ but $\mu = \mu_0$ holds very well

$$\vec{p}_{EM} = \frac{\vec{S}}{c^2} = \frac{1}{c^2}\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) \rightarrow \begin{cases} \epsilon_0 \mu_0 \dfrac{\vec{E}\times\vec{B}}{\mu_0} & \rightarrow \quad \vec{D}\times\vec{B} \\ \dfrac{1}{c^2}\vec{E}\times\dfrac{\vec{B}}{\mu_0} & \rightarrow \quad \dfrac{1}{c^2}\vec{E}\times\vec{H} \end{cases}$$

They are equivalent in vacuum but not in a medium !

Consider the plane monochromatic wave

$$\left.\begin{aligned} \vec{E} &= E_0 \hat{x} \cos\frac{2\pi}{\lambda}(z - vt) \\ \vec{H} &= H_0 \hat{y} \cos\frac{2\pi}{\lambda}(z - vt) \end{aligned}\right\} \qquad H_0 = \frac{E_0}{v\mu_0}$$

PHASE VELOCITY!

The momentum of light in a medium

$$u = \frac{1}{2}\left(\epsilon E^2 + \mu_0 H^2\right) = \frac{\epsilon}{2} E_0{}^2 \left\langle \cos^2 \frac{2\pi}{\lambda}(z - vt) \right\rangle \times 2$$

$$= \frac{1}{2}\epsilon E_0{}^2 = N\hbar\omega \Rightarrow \boxed{E_0{}^2 = \frac{2N\hbar\omega}{\epsilon}}$$

The energy is equally distributed in E and B fields

N is the number of photons per unit volume

$$\left\langle \vec{p}_{EM} \right\rangle = \left\langle \vec{D} \times \vec{B} \right\rangle = \epsilon E_0 \frac{E_0}{v} \left\langle \cos^2 \frac{2\pi}{\lambda}(z - vt) \right\rangle$$

$$= \epsilon \frac{E_0{}^2}{v} \frac{1}{2} = \frac{\epsilon}{v} \frac{2N\hbar\omega}{\epsilon} \frac{1}{2}$$

$$= \left(\frac{c}{v}\right) \frac{N\hbar\omega}{c} = \boxed{n\left(\frac{N\hbar\omega}{c}\right)}$$

proposed by Minkowski (1908)

The momentum of light in a medium

$$\langle \vec{p}_{EM} \rangle = \frac{1}{c^2}\langle \vec{E}\times\vec{H}\rangle = \epsilon_0 \mu_0 E_0 \frac{E_0}{v\mu_0}\left\langle \cos^2 \frac{2\pi}{\lambda}(z-vt)\right\rangle$$

$$= \epsilon_0 \frac{E_0^{\ 2}}{v}\frac{1}{2} = \frac{\epsilon_0}{v}\frac{2N\hbar\omega}{\epsilon}\frac{1}{2}$$

$$= \left(\frac{\epsilon_0}{\epsilon}\right) n \frac{N\hbar\omega}{c} = \frac{1}{n}\left(\frac{N\hbar\omega}{c}\right)$$

proposed by Abraham (1909)

The difference actually points to the limitation of the macroscopic description of a medium composed of discrete atoms.

At the atomic level the electromagnetic field and atomic motion is mixed up inseparably. One must consider the EM field + atoms system and write out the expression for total momentum.

The "Abraham" or "Minkowski“ result makes sense only if the "matching piece" of atomic motion is included.

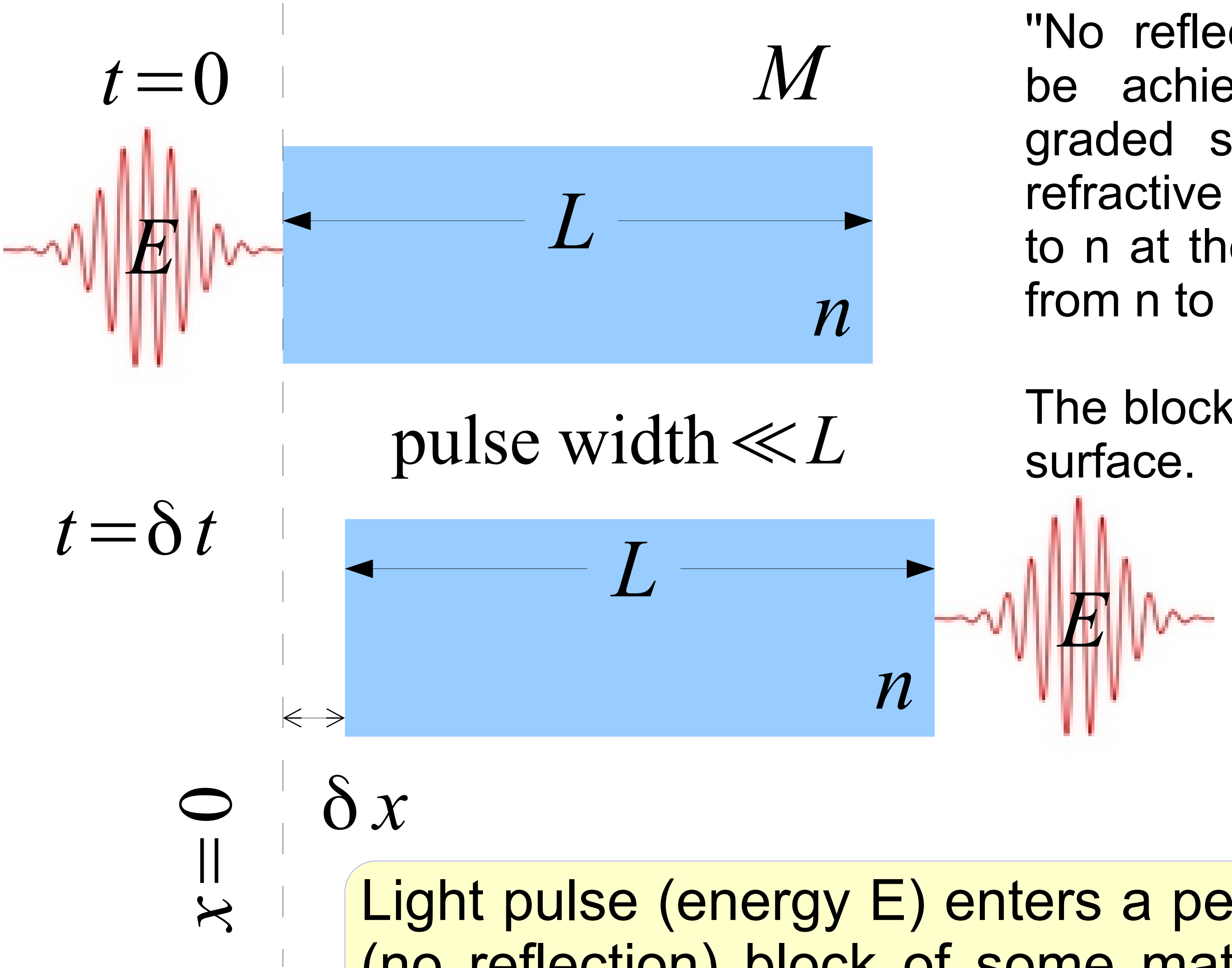
Semiclassically the same question can be asked

Viewpoint 1: The speed of light has been slowed down by a factor of n (the refractive index) .So momentum will REDUCE by a factor of n, like that of any classical "particle"

Viewpoint 2: The wavelength of light has changed from λ to λ/n. So the wavevector (k) must have become LARGER in magnitude. We know that momentum is proportional to the wavevector, hence momentum should INCREASE ?

Question : Can one design an experiment to ask this question ? Simply immersing a mirror in a "medium" and measuring recoil does not answer this question – because atoms of the medium would keep hitting the mirror all the time!

A thought experiment



"No reflection" condition can be achieved by making a graded structure, where the refractive index varies from 1 to n at the left edge and then from n to 1 at the right edge.

The block sits on a frictionless surface.

Proposed by Balazs (1953)

Light pulse (energy E) enters a perfectly transparent (no reflection) block of some material at t=0 and leaves the block at t=δt . By how much does the block move? Should it go forward or backward?

A thought experiment

The answer depends on the momentum of the pulse when it was inside the block. Why ?

$$p_{\text{out}} = Mv + p_{\text{in}}$$

$$\delta x = \frac{p_{\text{out}} - p_{\text{in}}}{M}\left(\frac{L + \delta x}{c/n}\right)$$

Block starts moving when pulse enters becuase total momentum must be conserved.

Block stops moving when pulse leaves because light takes away all the momentum

The position of the leading edge of the block when pulse leaves the block.

$$\delta x = \frac{Ln}{Mc}\left(\frac{E}{c} - p_{\text{in}}\right)$$

So measuring the displacement would tell what p_{in} was ?

Question : Can the block move rigidly on such timescales?

Momentum of light in a medium : references

References for some in-depth discussion.. Notice that the papers are quite recent compared to how long Maxwell's equations have been around!

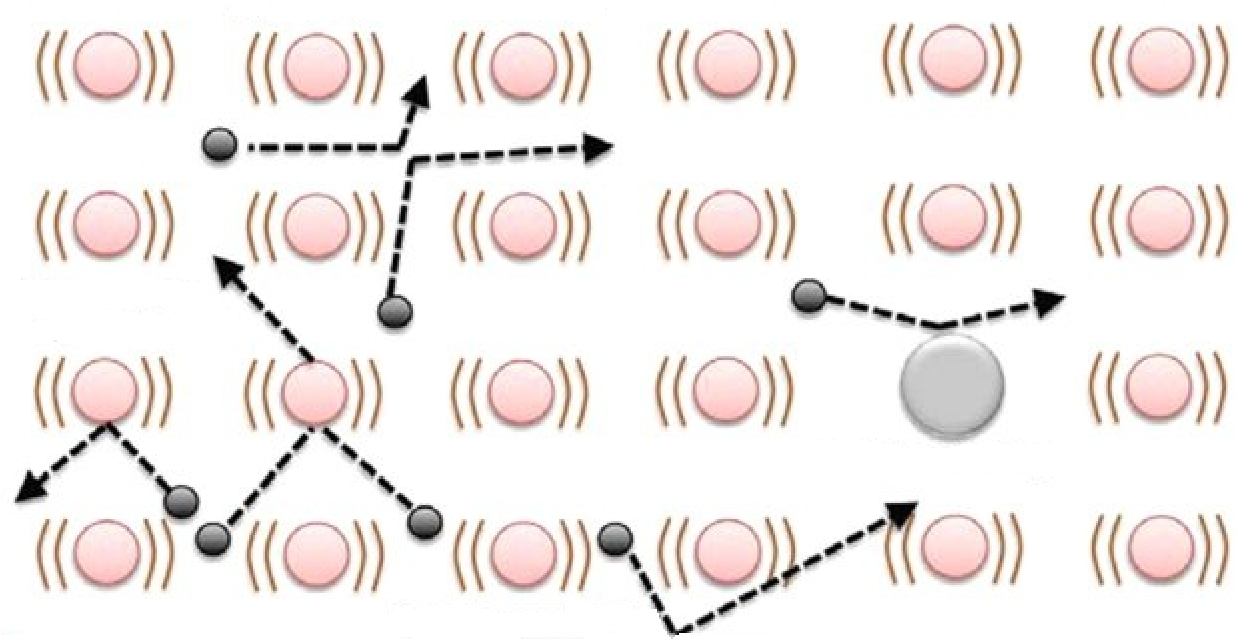
Momentum of Light in a Dielectric Medium
Peter W. Milonni and Robert W. Boyd
Advances in Optics and Photonics **2**, 519–553 (2010)

***Colloquium*: Momentum of an electromagnetic wave in dielectric media**
Robert N. C. Pfeifer, Timo A. Nieminen, Norman R. Heckenberg, and Halina Rubinsztein-Dunlop
Reviews of Modern Physics, **79**, 1197
Erratum: *Colloquium*: Momentum of an electromagnetic wave in dielectric media 79, 1197 (2007) in vol **81** Jan 2009 issue

The enigma of optical momentum in a medium
Stephen M. Barnett and Rodney Loudon
Phil. Trans. R. Soc. ***A* (2010) 368**, 927–939

Note: These have a lot of detail and descriptions of the experiments tried to answer the question. These are for additional reading (not exam syllabus)

A bit of the microscopic picture



The lattice & the "free electrons“ bouncing around

How the two types of electrons (bound to atoms + free) respond to a field determine what the dielectric function will be.

In reality "bound" and "free" are two extremes. There can intermediates. But this will illustrate two important types of behaviour.

Forced oscillation of the bound electrons

Important : Wavelength is such that kx varies very little over the length scale of interest. Ok for an atom/molecule & light.....
But not for hard Xray, gamma ray etc!

$$E = E_0 \cos(kx - \omega t) \approx \Re\left(E_0 e^{-i\omega t}\right)$$

$$M\ddot{x} = -b\dot{x} - kx + qE_0 \cos(kx - \omega t)$$

$$x(t) = \Re\,\tilde{x}_0 e^{-i\omega t} \qquad \left\{\gamma = \frac{b}{M} \;\;\&\;\; \omega_0^2 = \frac{k}{M}\right\}$$

$$\tilde{x}_0 = \frac{q/M}{(\omega_0^2 - \omega^2) - i\gamma\omega} E_0$$

$$Nq\tilde{x}_0 = \epsilon_0 \left[\frac{Nq^2/M\epsilon_0}{(\omega_0^2 - \omega^2) - i\gamma\omega}\right] E_0 \Rightarrow \boxed{\tilde{P} = \epsilon_0 \tilde{\chi}(\omega) E_0}$$

Getting the dispersion [$\omega(k)$ relation]

$$\tilde{\epsilon}(\omega) = \epsilon_0 \left[1 + \frac{Nq^2}{M\epsilon_0} \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} e^{-i\phi} \right]$$

$$\tan\phi = \frac{\gamma\omega}{\omega_0^2-\omega^2} = \begin{cases} \approx 0 & \text{for } \omega \ll \omega_0 \\ \approx \frac{\pi}{2} & \text{for } \omega \approx \omega_0 \\ \rightarrow \pi & \text{for } \omega \gg \omega_0 \end{cases}$$

The sign change etc. are characteristic of any resonant response
But to get the dispersion we need to solve

$$\nabla^2 \tilde{\vec{E}} = \tilde{\epsilon}\mu_0 \frac{\partial^2 \tilde{\vec{E}}}{\partial t^2}$$

wavevector must become complex too

Significance of the real and imaginary parts of k

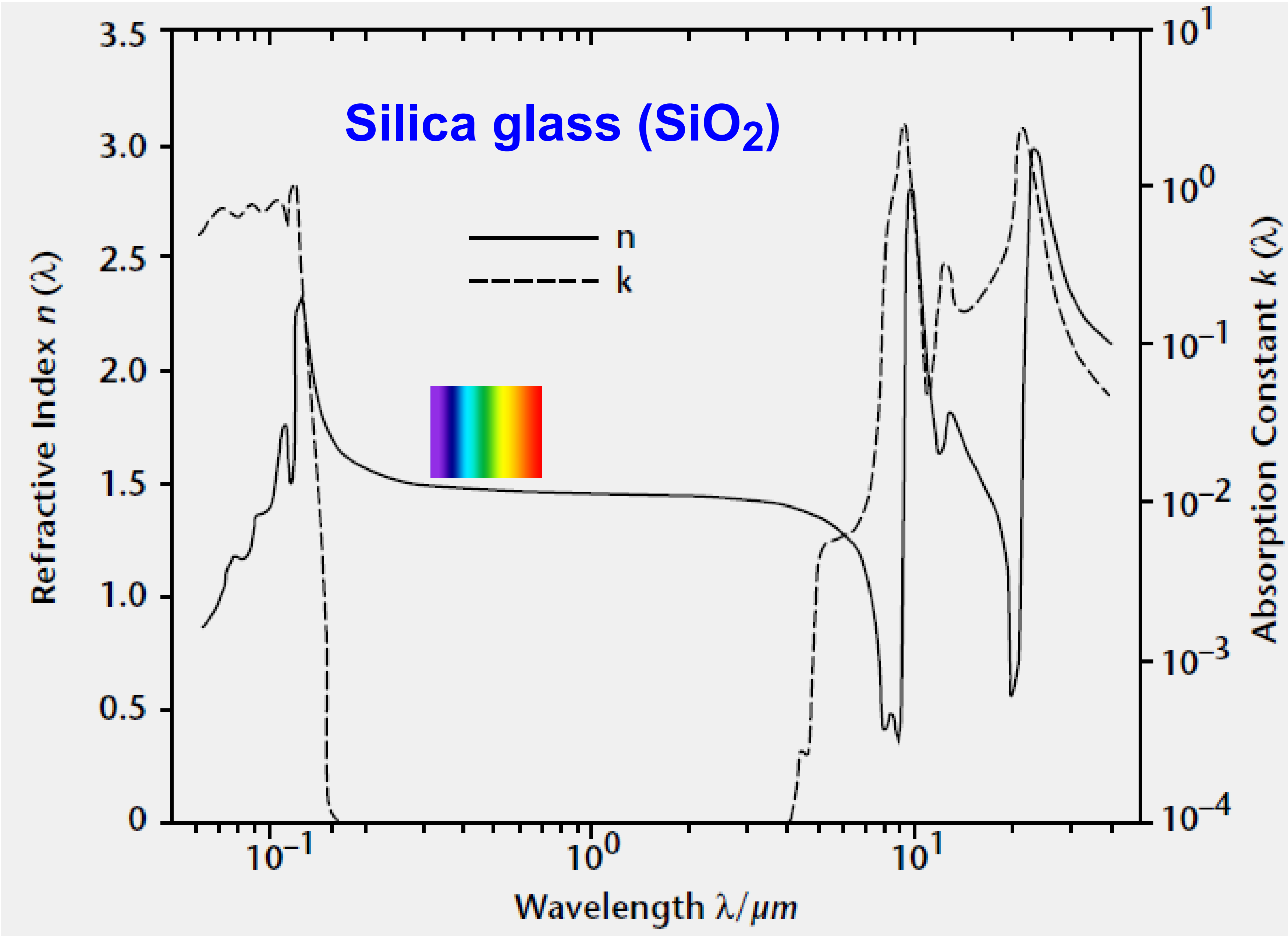
$$\tilde{E}(x,t) = E_0 \exp\left[i(k'+ik'')x-\omega t\right]$$

$$= E_0 \, e^{-k''x} \, \exp\left[i(k'x-\omega t)\right]$$

Refractive index	$n(\omega)$	$=$	$\frac{c}{\omega}\ \mathrm{Re}\ (k)$
Absorption coefficient	$\alpha(\omega)$	$=$	$2\ \mathrm{Im}(k)$

In reality many resonance are scattered all over the spectrum for a real material. Since there are various kinds of atoms, bondings etc that are involved. We cannot write generalised or explicit solutions any more, but the origin of the variations are qualitateively explained by forced and moderately damped vibrations of the bound electrons.

An example of the variation



When there is no restoring force

$$nq\,\tilde{x}(t) = \frac{nq^2/m}{(\cancel{\omega_0^2}-\omega^2)-i\gamma\omega}\,E_0 e^{-i\omega t}$$

$$\frac{d}{dt}\left(nq\,\tilde{x}(t)\right) = \frac{nq^2}{m}\,\frac{-i\omega}{-\omega^2-i\omega/\tau}\,E_0 e^{-i\omega t} \qquad (\gamma\to 1/\tau)$$

$$\tilde{j}_{free}(\omega) = \boxed{\frac{n\,q^2\,\tau}{m}}\,\frac{1}{1-i\omega\tau}\,E_0 e^{-i\omega t} = \frac{\boxed{\sigma_0}}{1-i\omega\tau}\,E_0 e^{-i\omega t}$$

This is the conventional "Drude" expression of the current, with dissipation set equal to inverse of "relaxation time".

Now the job is to get the total polarisation including the contribution of the free electrons and the lattice.

Adding free & bound contributions

$$\begin{aligned}
\tilde{P}_{tot} &= \tilde{P}_{free} + \tilde{P}_{bound} \\
\frac{d\tilde{P}_{tot}}{dt} &= \frac{d}{dt} n_{free} q \tilde{u} + \frac{d\tilde{P}_b}{dt} \\
\epsilon_0 \chi(\omega)(-i\omega)\tilde{E}_0 &= \sigma(\omega)\tilde{E}_0 + \epsilon_0 \chi_b(\omega)(-i\omega)\tilde{E}_0 \\
\frac{\sigma(\omega)}{\epsilon_0} &= i\omega(\chi_b - \chi_{tot}) \\
\chi_{tot}(\omega) &= \chi_b + i\frac{\sigma(\omega)}{\omega\epsilon_0} \\
\epsilon_{tot}(\omega) &= \epsilon_b + i\frac{\sigma(\omega)}{\omega} \\
\epsilon_{tot}(\omega) &= \epsilon_b + i\epsilon_0 \frac{\omega_p^2}{\omega(1/\tau - i\omega)}
\end{aligned}$$

Plasma freq:

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$$

This relation between $\epsilon(\omega)$ and $\sigma(\omega)$ is used in many forms.

The role of Plasma frequency

$$n^2 = \frac{\epsilon_{tot}}{\epsilon_0} = \frac{\epsilon_b}{\epsilon_0} + i\frac{{\omega_p}^2}{\omega(1/\tau - i\omega)}$$

$$\tau \sim 10^{-14}\,\mathrm{sec}\,,\ \omega_p \sim 10^{16}\,\mathrm{Hz}$$

Noble metal	Electron density ($10^{22}/\mathrm{cm}^3$)	Plasma frequency (10^{16} Hz)
Gold (Au)	5.90	1.40
Silver (Ag)	5.86	1.39
Copper (Cu)	8.47	1.64

The second term will dominate when $\omega < \omega_p$

The expression will again become almost real when $\omega \gg \omega_p$

Waves will propagate through plasma when $\omega > \omega_p$ with some loss

But at lower frequencies there can be near perfect reflection

Propagation at normal incidence

$$\vec{E}_I = \hat{x}\, E_I\, e^{i(kz+\omega t)}$$

$$\vec{H}_I = -\hat{y}\,\frac{E_I}{c}\, e^{i(kz+\omega t)}$$

$$\vec{E}_T = \hat{x}\, E_T\, e^{i(nkz+\omega t)}$$

$$\vec{H}_T = -\hat{y}\,\frac{E_T}{c/n}\, e^{i(nkz+\omega t)}$$

$$\vec{E}_R = \hat{x}\, E_R\, e^{i(kz-\omega t)}$$

$$\vec{H}_R = \hat{y}\,\frac{E_R}{c}\, e^{i(kz-\omega t)}$$

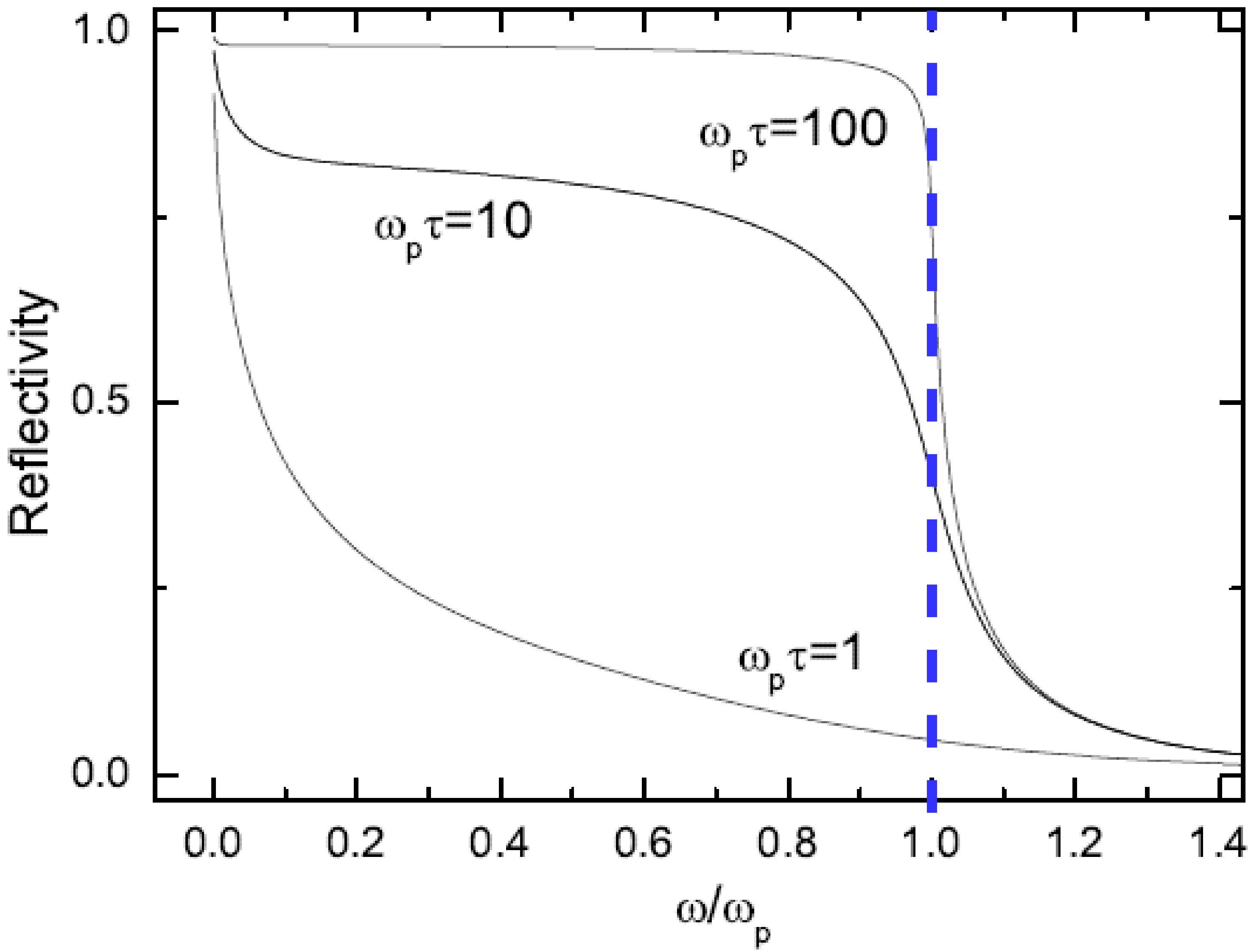
We assume the form of E and then calculate what H must be from the Maxwell's equations. It can be done otherwise also of course.

$E_{\parallel}$ and $H_{\parallel}$ are continuous $\Rightarrow$

$$\begin{cases} E_I + E_R = E_T \\ -E_I + E_R = -nE_T \end{cases}$$

Calculated from the derived relation...

$$\left|\frac{E_R}{E_I}\right|^2 = \left|\frac{1-n}{1+n}\right|^2 \quad \& \quad \left|\frac{E_T}{E_I}\right|^2 = \left|\frac{2}{1+n}\right|^2$$

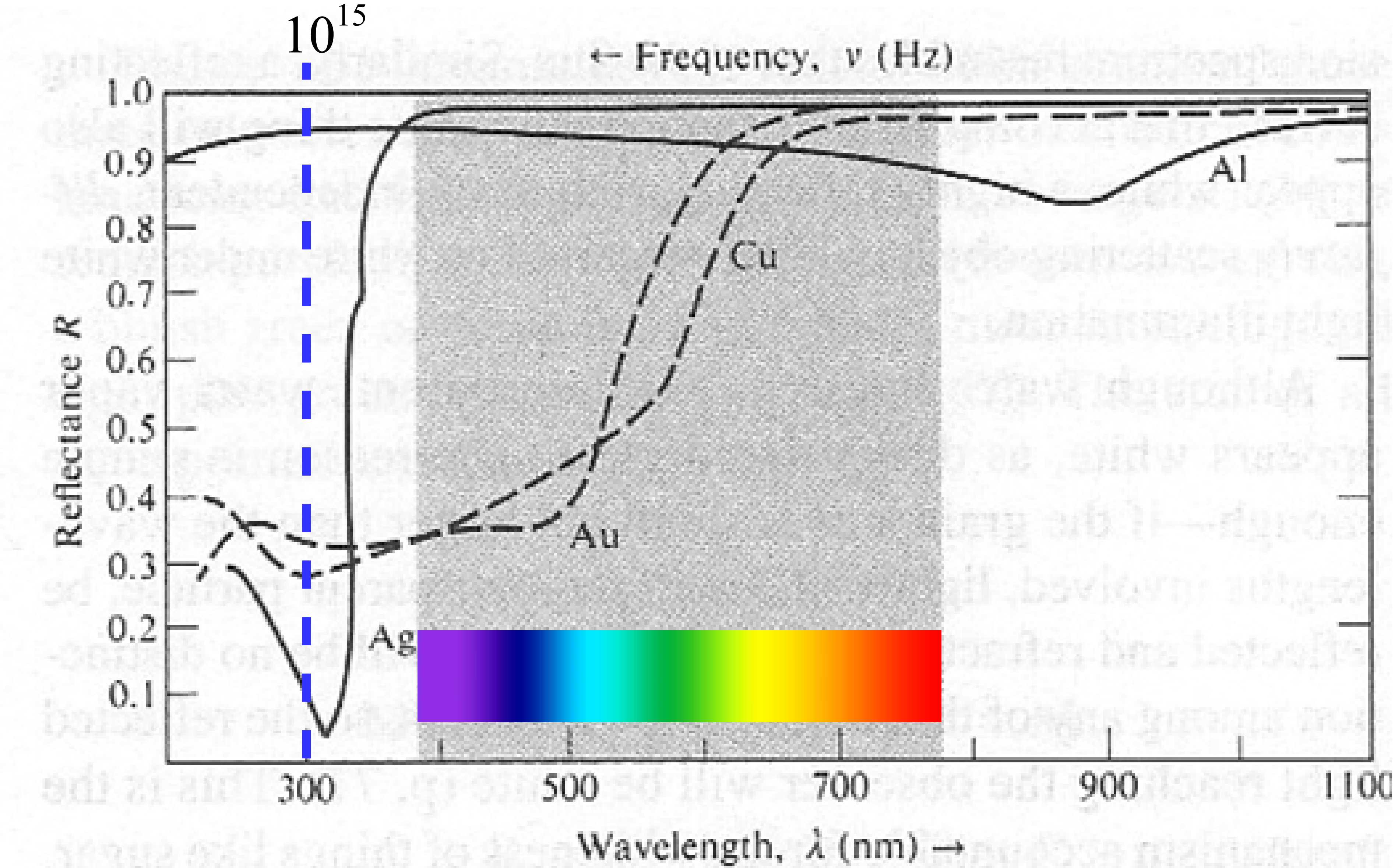


Notice that a good metal reflects almost everything below a critical frequency. That is why it is shiny.

The wave rapidly decays inside the the region where there are free electrons. A metal and a plasma behave in similar ways.

Plasma frequency of the ionosphere is several Mhz typically

Variation of reflectance with λ



To compare with the last figure notice that the axis here uses wavelength, not frequency, so the curve is flipped left-right.

Reflection from ionosphere

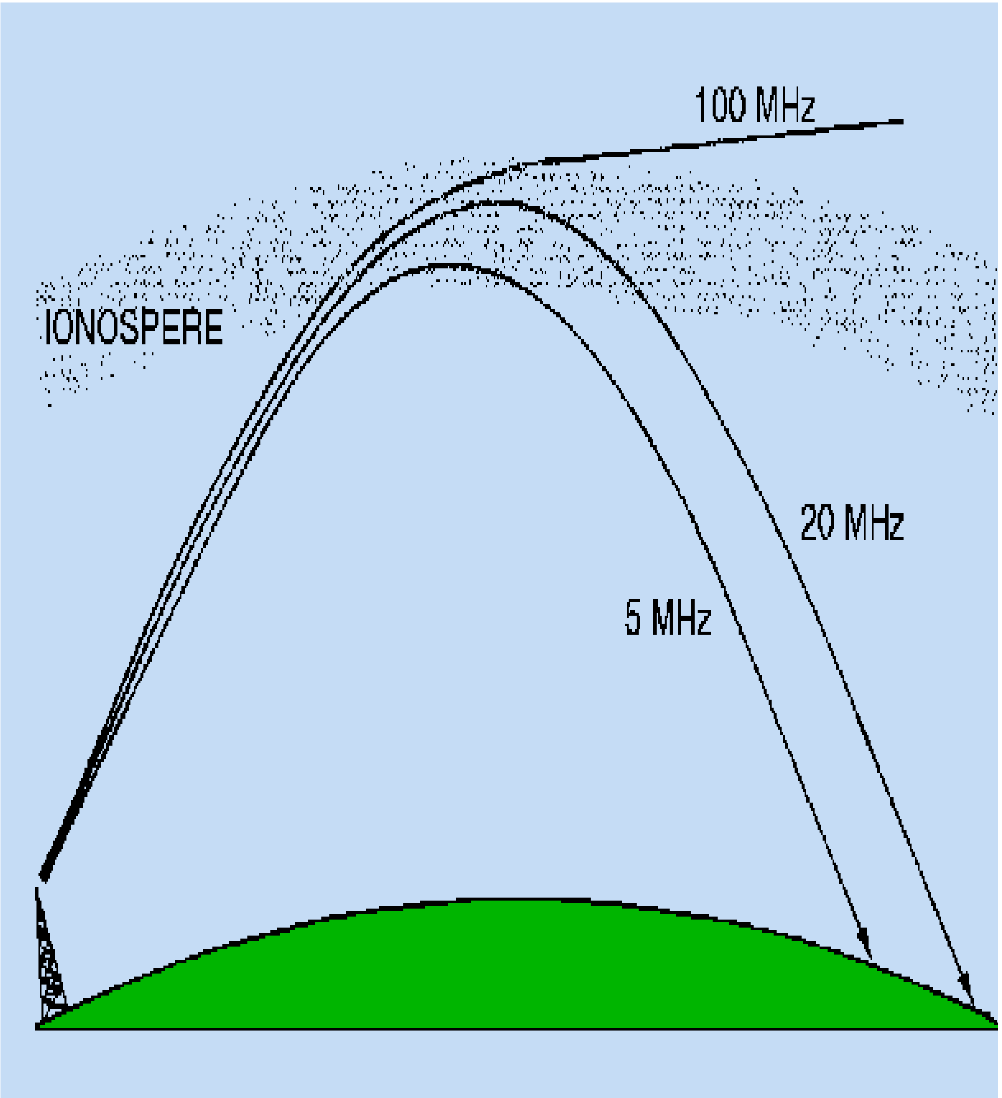
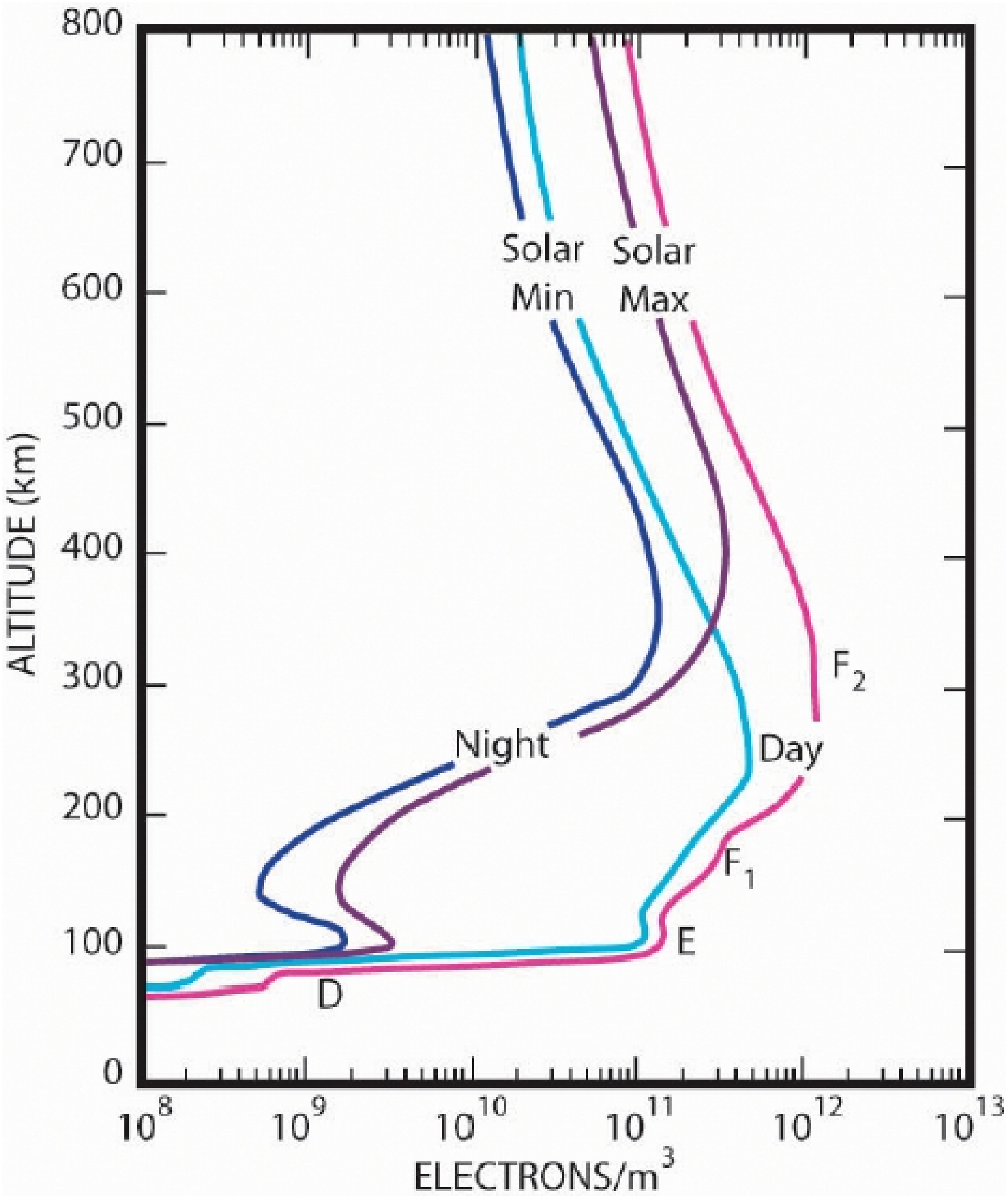


Image : http://www.tpub.com/neets/book10/40e.htm



Note the very different carrier density. $\omega_p \propto \sqrt{n}$

Reflection, refraction, evanescent waves

Reflection and refraction of light at an interface

Consider a boundary between two media 1 and 2
div D = 0, → normal component of D must be continuous.
div B = 0, always (so normal component of B is continuous)
curl H has no singularities → tangential component of H is continuous
curl E has no singularities …tangnetial component of E is continuous

$$D_1^{\perp} = D_2^{\perp} \qquad Hence \quad \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$
$$B_1^{\perp} = B_2^{\perp}$$
$$H_1^{\parallel} = H_2^{\parallel} \qquad Hence \quad \frac{B_1^{\parallel}}{\mu_1} = \frac{B_2^{\parallel}}{\mu_2}$$
$$E_1^{\parallel} = E_2^{\parallel}$$

These boundary conditions govern the reflection and transmission of electromagnetic waves at an interface and hence the laws of reflection and refraction (optics)

Why can the frequency not change ?

$$A e^{iax} + B e^{ibx} = C e^{icx} \quad \forall \quad x$$

$$\textit{Then} \quad a = b = c$$

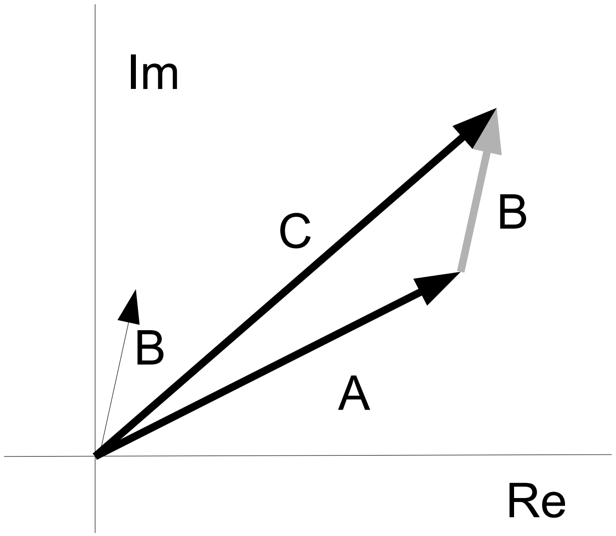
$$\textit{set } x = 0 \quad : \quad \textit{this gives} \quad A + B = C$$

$$\textit{Now draw the three phasors when} \quad x \neq 0$$

Two sides of a traingle are together greater than the third side

The equality can only hold if A, B, C *are along the same ray..*
The phase angle also must be same implies $a = b = c$

This condition determines the length of the phasors, which must be satisfied at all times

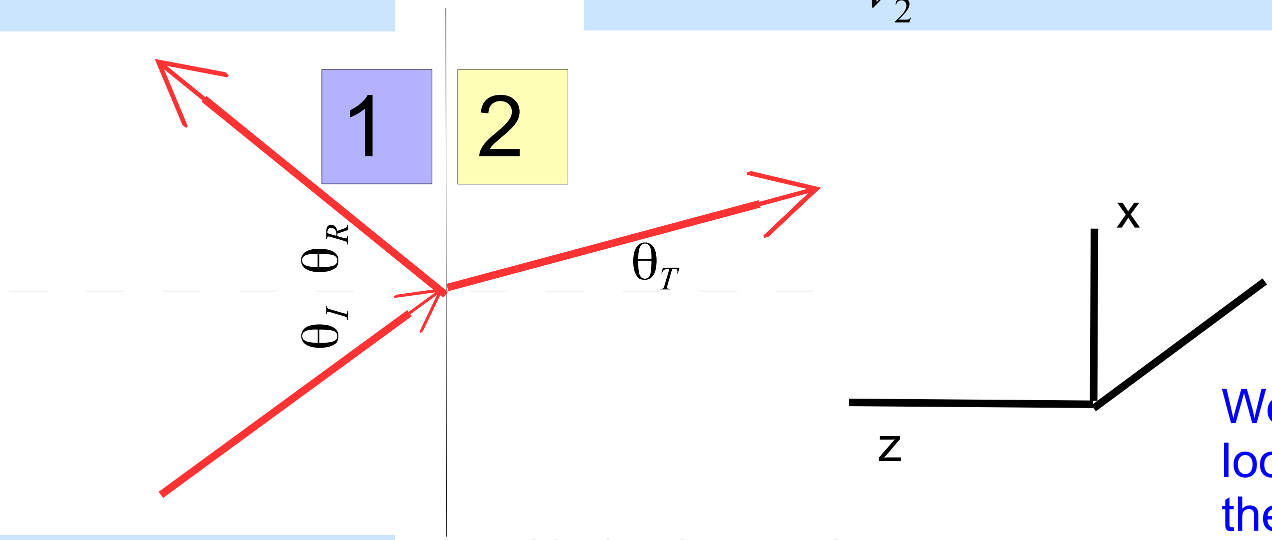


Then identify x → t and a,b,c → ω

The incident reflected and transmitted waves

$$\vec{E}_R = \vec{E}_{0R} \exp\left[i\left(\vec{k}_R . \vec{r} - \omega t\right)\right]$$
$$\vec{B}_R = \frac{\hat{k}_R \times \vec{E}_{0R}}{v_1} \exp\left[i\left(\vec{k}_R . \vec{r} - \omega t\right)\right]$$

$$\vec{E}_T = \vec{E}_{0T} \exp\left[i\left(\vec{k}_T . \vec{r} - \omega t\right)\right]$$
$$\vec{B}_T = \frac{\hat{k}_T \times \vec{E}_{0T}}{v_2} \exp\left[i\left(\vec{k}_T . \vec{r} - \omega t\right)\right]$$



We are looking at the z=0 plane sideways

$$\vec{E}_I = \vec{E}_{0I} \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right]$$
$$\vec{B}_I = \frac{\hat{k}_I \times \vec{E}_{0I}}{v_1} \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right]$$

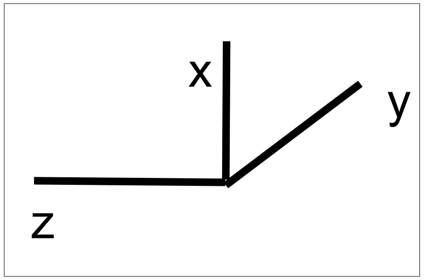
Notice how unit vectors have been used to fix the relative directions

$$\omega = |\vec{k}| v \quad : \quad Hence \quad k_I v_1 = k_R v_1 = k_T v_2$$
$$\vec{k}_I . \vec{r} = \vec{k}_R . \vec{r} = \vec{k}_T . \vec{r} \quad must \quad hold \quad \forall r \quad \text{on } the\ z=0\ plane$$

The laws of reflection and refraction

$$k_I = k_R = \frac{v_2}{v_1} k_T \quad \text{in magnitude}$$

$$\left.\begin{aligned} (k_I)_x x + (k_I)_y y &= (k_R)_x x + (k_R)_y y \\ (k_I)_x x + (k_I)_y y &= (k_T)_x x + (k_T)_y y \end{aligned}\right\} \quad \forall \quad x, y$$



The coefficients (y,z components) must be equal

$$\vec{k}_I . \vec{k}_R \times \vec{k}_T = 0$$ since two row/columns are identical.

The three vectors are co-planer [Law of reflection and refraction]
In this case it is the x-z plane.

Since $|\boldsymbol{k_I}| = |\boldsymbol{k_R}|$ and y components are equal, the other (z) component is exactly reversed.
No other possibility can satisfy all these conditions.

The laws of reflection and refraction

Consider the x component

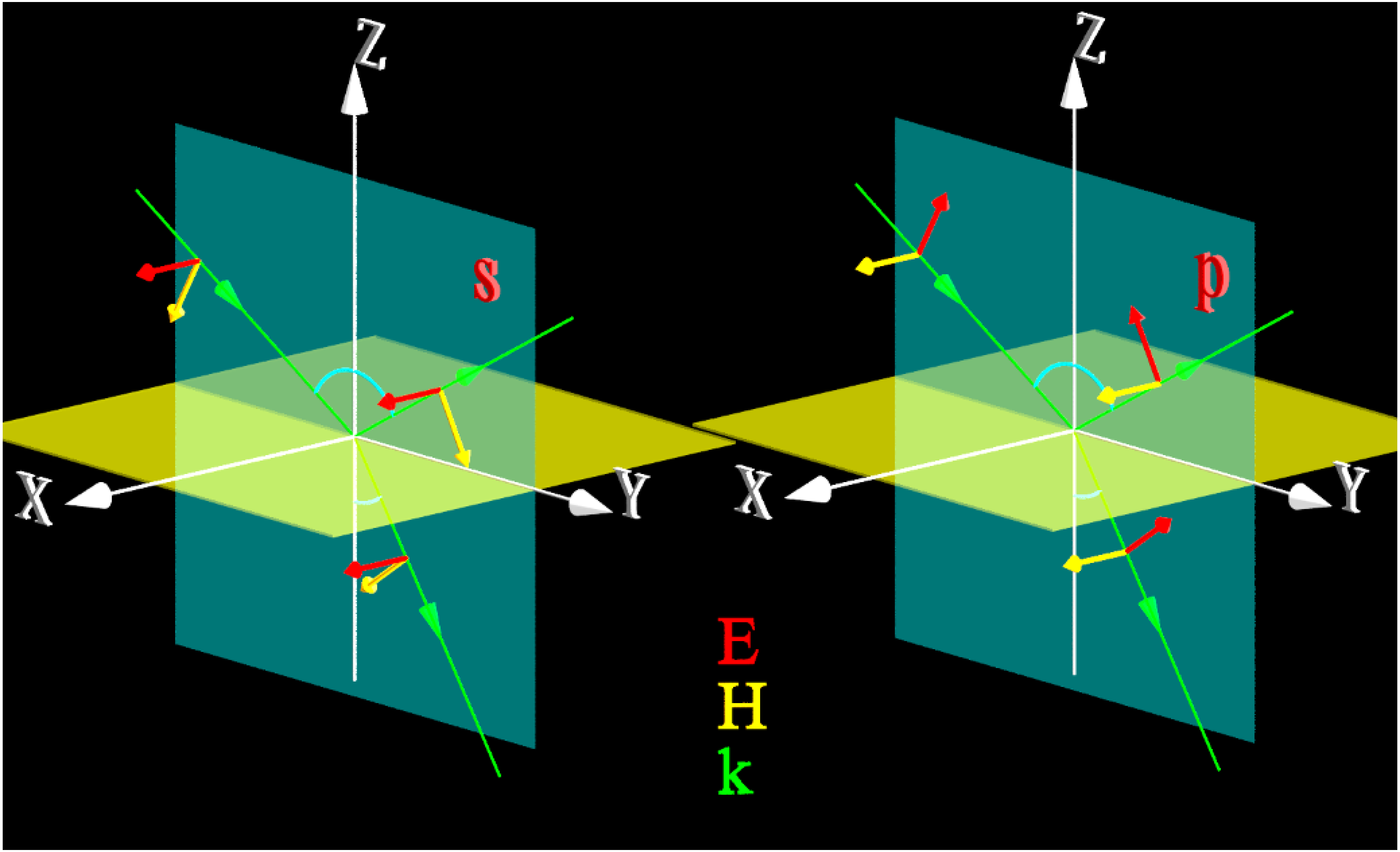
$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

$$\theta_I = \theta_R$$

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

We haven't really used the boundary conditions so far.
Only their format is sufficient to establish Snell's law !

Now generalise the problem....



The s (σ) polarisation of the incident ray

$$
\begin{aligned}
\vec{k}_I &= k_I \sin\theta_I \hat{y} - k_I \cos\theta_I \hat{z} \\
\vec{k}_R &= k_R \sin\theta_I \hat{y} + k_R \cos\theta_I \hat{z} \\
\vec{k}_T &= k_T \sin\theta_T \hat{y} - k_T \cos\theta_T \hat{z}
\end{aligned}
$$

$$
\begin{aligned}
\vec{E}_I &= E_{0\mathrm{I}} \hat{x} \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right] \\
\vec{E}_R &= E_{0\mathrm{R}} \hat{x} \exp\left[i\left(\vec{k}_R . \vec{r} - \omega t\right)\right] \\
\vec{E}_T &= E_{0\mathrm{T}} \hat{x} \exp\left[i\left(\vec{k}_T . \vec{r} - \omega t\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \vec{E} &= -\mu_1 \frac{\partial \vec{H}}{\partial t} \\
\nabla \times \vec{H} &= \epsilon_1 \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{H}_I &= -\frac{E_{0\mathrm{I}}}{v_1 \mu_1}\left(\cos\theta_I \hat{y} + \sin\theta_I \hat{z}\right) \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right] \\
\vec{H}_R &= -\frac{E_{0\mathrm{R}}}{v_1 \mu_1}\left(\cos\theta_I \hat{y} - \sin\theta_I \hat{z}\right) \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right] \\
\vec{H}_T &= -\frac{E_{0\mathrm{T}}}{v_2 \mu_2}\left(\cos\theta_T \hat{y} + \sin\theta_T \hat{z}\right) \exp\left[i\left(\vec{k}_I . \vec{r} - \omega t\right)\right]
\end{aligned}
$$

Solve for E_{0R} and E_{0T} in terms of E_{0I}

There is no normal component of E → three sets of equations.
Of these one would just reproduce Snell's law
Use the other two to do the job :

$$\frac{E_{0T}}{E_{0I}} = \frac{2\mu_2 v_2 \cos\theta_I}{\mu_2 v_2 \cos\theta_I + \mu_1 v_1 \cos\theta_T}$$

$$\frac{E_{0R}}{E_{0I}} = \frac{\mu_2 v_2 \cos\theta_I - \mu_1 v_1 \cos\theta_T}{\mu_2 v_2 \cos\theta_I + \mu_1 v_1 \cos\theta_T}$$

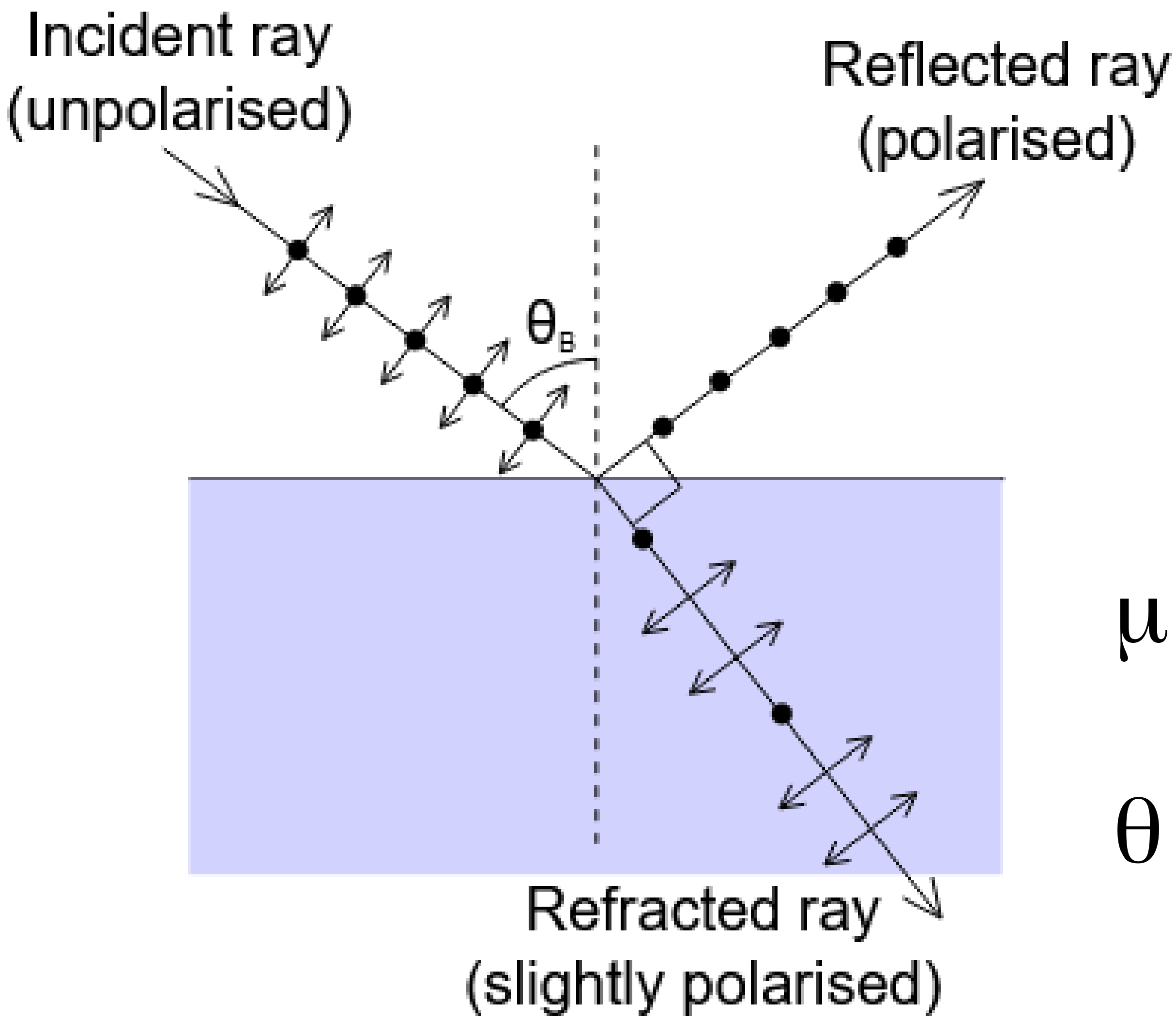
$\boxed{\sigma}$

$$\frac{E_{0T}}{E_{0I}} = \frac{2\mu_2 v_2 \cos\theta_I}{\mu_2 v_2 \cos\theta_T + \mu_1 v_1 \cos\theta_I}$$

$$\frac{E_{0R}}{E_{0I}} = \frac{\mu_2 v_2 \cos\theta_T - \mu_1 v_1 \cos\theta_I}{\mu_2 v_2 \cos\theta_T + \mu_1 v_1 \cos\theta_I}$$

$\boxed{\pi}$

Full transmission of π pol : Brewster angle

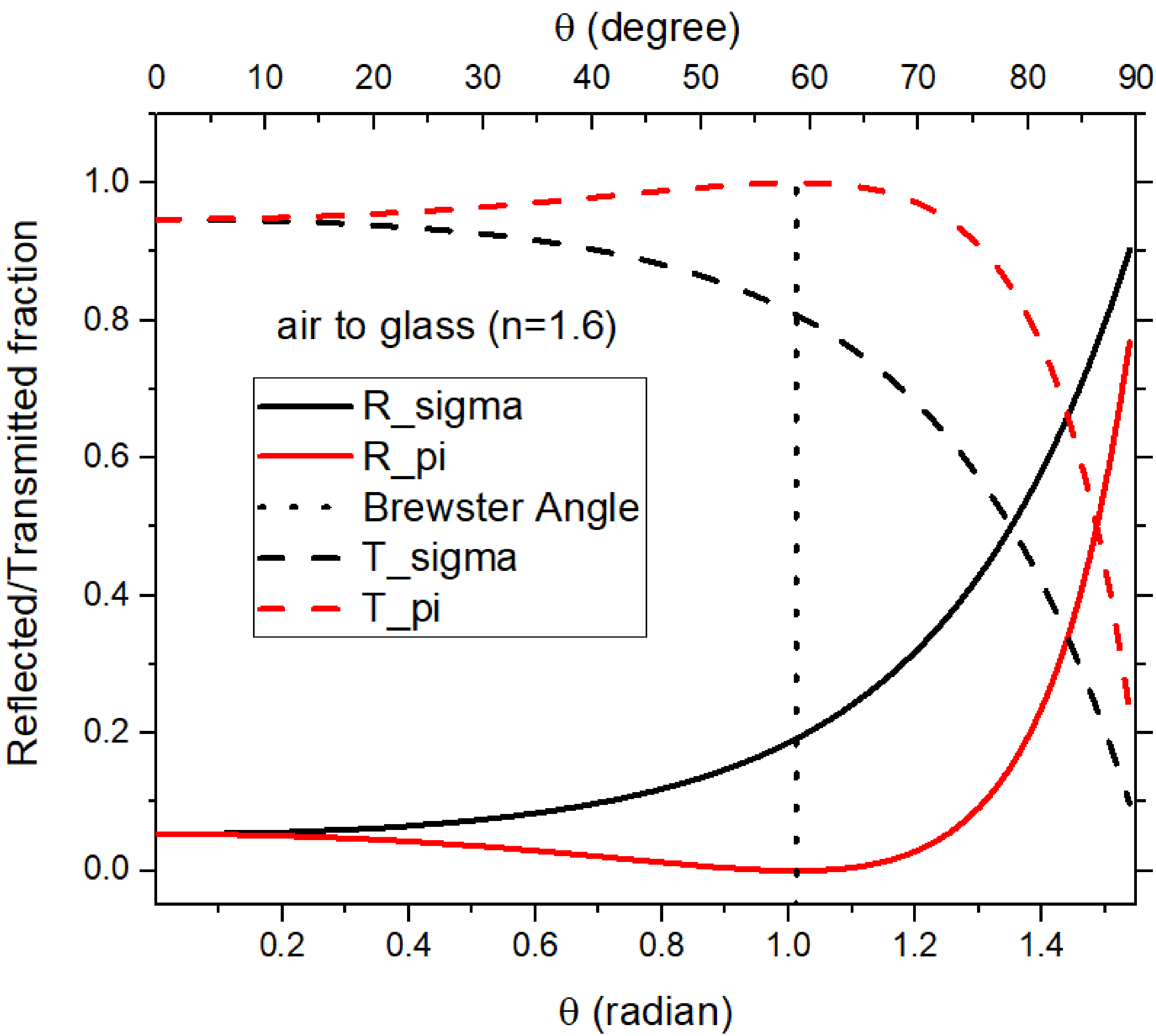


$$\mu_1 = \mu_2$$

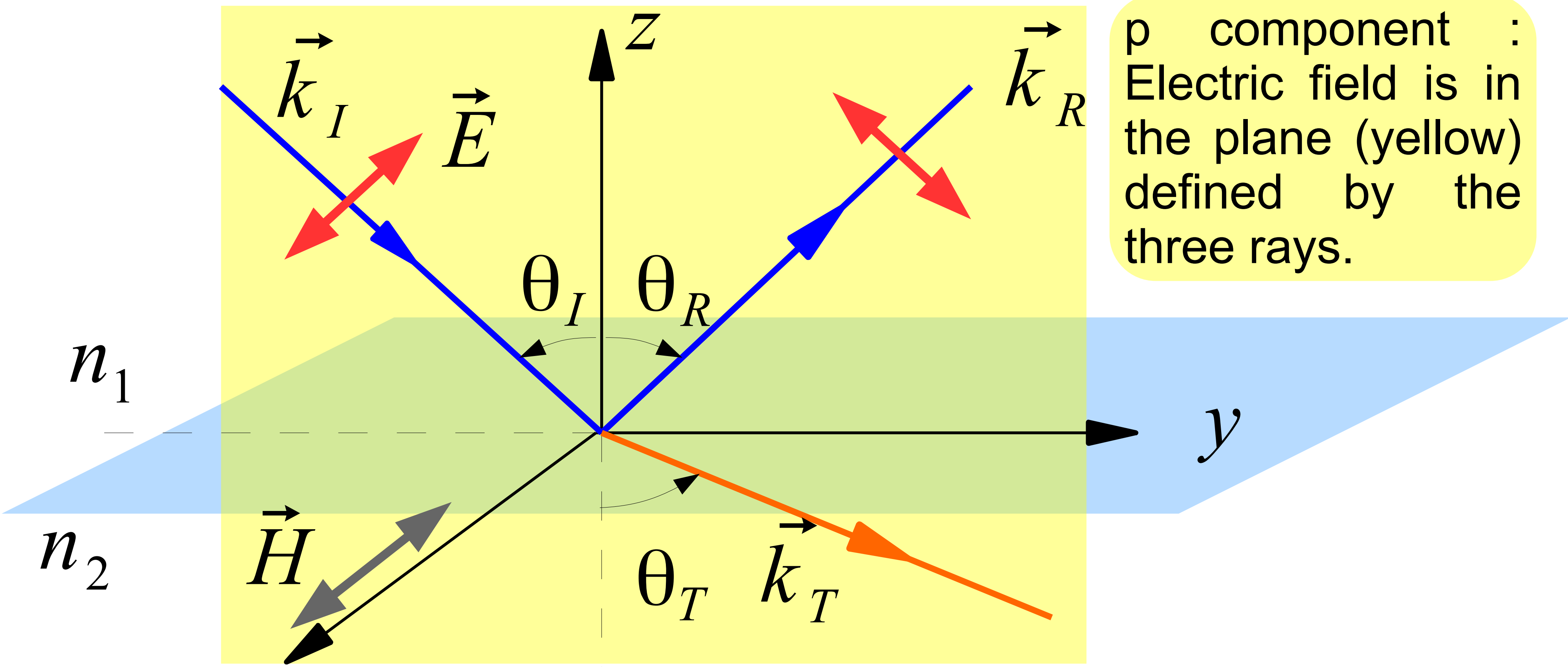
$$\theta = \arctan\left(\frac{n_2}{n_1}\right)$$

https://commons.wikimedia.org/w/index.php?curid=2519325

Reflected and transmitted fractions



Evanescent waves and total internal reflection



p component : Electric field is in the plane (yellow) defined by the three rays.

$$k_0 = \frac{\omega}{c} \Rightarrow \begin{cases} \vec{k}_I = k_0\left(\hat{y}\, n_1 \sin\theta_I - \hat{z}\, n_1 \cos\theta_I\right) \\ \vec{k}_R = k_0\left(\hat{y}\, n_1 \sin\theta_R + \hat{z}\, n_1 \cos\theta_R\right) \\ \vec{k}_T = k_0\left(\hat{y}\, n_2 \sin\theta_T - \hat{z}\, n_2 \cos\theta_T\right) \end{cases}$$

The necessity of having evanescent field

In total internal reflection we often say that the reflection is "complete"

But if E,B, are all zero in the other "rarer" medium it would be impossible to match the boundary condition as we just discussed.

This is true irrespective of which polarisation (s, p) the incident wave might have.

Thus something must exist in the second medium too!

Evanescent wave

In medium $2: \sin\theta_T = \frac{n_1}{n_2}\sin\theta_I \qquad (n_1 > n_2 \; : \; \text{TIR possible})$

$$\vec{E}_T = \vec{E}_{T0}\exp\left[i\left(k_0 n_2 \sin\theta_T\, y - k_0 n_2 \cos\theta_T\, z - \omega t\right)\right]$$
$$= \vec{E}_{T0}\exp\left[i\left(k_0 n_1 \sin\theta_I\, y - k_0 n_2 \cos\theta_T\, z - \omega t\right)\right]$$

$$\cos\theta_T = \pm\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I}$$

Total internal reflection implies that this will be > 1

$$n_2\cos\theta_T = \pm i\sqrt{n_1^2\sin^2\theta_I - n_2^2}$$

Which sign should be chosen ?

$$\vec{E}_T = \vec{E}_{T0}\, e^{\left(k_0\sqrt{n_1^2\sin^2\theta_I - n_2^2}\right) z}\, e^{i\left(k_0 n_1 \sin\theta_I\, y - \omega t\right)}$$

damped as $z \to -\infty$ | propagates along y

Evanescent wave

The solution can be used to write E,D and B,H using Maxwell's equation.

Since we have got solutions in both regions we can match the boundary conditions. EM boundary conditions must hold irrespective of whether the reflection is total or partial.

The wave propagates with the same wavevector it had in the other medium.

For practical values of refractive indices the wave will penetrate for 1-2 wavelengths only.

If a detector or another interface is brought within this distance it will sense the fields of the evanescent mode.

Evanescent wave

References : D.J. Griffiths, chapter 8 (EM waves)

Wikipedia articles on Frsenel equations and Total internal reflection are both very good:

https://en.wikipedia.org/wiki/Fresnel_equations#Theory

https://en.wikipedia.org/wiki/Total_internal_reflection#Evanescent_wave_(qualitative_explanation)

Potential formulation, moving charges and radiation

1. Potentials and gauge
2. Retarded potential
3. Point dipole and half wave antenna
4. Moving point charge (Leinard Wiechart factor)
5. Uniformly moving point charge
6. Accelerated point charge
7. Brehmstralung and Synchrotron radiation
8. Čerenkov radiation
9. Radiation retardation

The time dependent potential formulation

$$\nabla . \vec{B} = 0 \quad \Rightarrow \quad \boxed{\vec{B} = \nabla \times \vec{A}} \quad \text{always possible}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V} \qquad \text{The potential}$$

$$\nabla . \vec{E} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \boxed{\nabla^2 V + \frac{\partial}{\partial t}\left(\nabla . \vec{A} \right) = -\frac{\rho}{\epsilon_0}}$$

The choice div. A = 0 leads to the Possion's equation. Poisson's equation has no time dependence into it. This implies that if the charge density changes, the potential must change instantaneously at all points. This cannot be correct in dynamic situations.

Q: What condition will relate dependent J and A ?

The time dependent potential formulation

$$\nabla\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\nabla\times\vec{A} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{A}}{\partial t} - \nabla V\right)$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \nabla\left(\nabla.\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right)$$

The choice

$$\nabla.\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \Rightarrow \begin{cases} \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \\[2ex] \nabla^2 V - \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\epsilon_0} \end{cases}$$

called Lorentz gauge

How to solve this in the Lorentz gauge

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

Use the Fourier transform method

$$\left.\begin{aligned} \rho(\vec{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\rho}(\vec{r},\omega)e^{-i\omega t}\,d\omega \\ V(\vec{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{V}(\vec{r},\omega)e^{-i\omega t}\,d\omega \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} \nabla^2\tilde{V} + \frac{\omega^2}{c^2}\tilde{V} &= -\frac{\tilde{\rho}}{\epsilon_0} \\ &= -\tilde{g} \end{aligned}$$

Now solve for the Green's function

$$\nabla^2 G(\vec{r},\vec{r}') + \frac{\omega^2}{c^2}G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}')$$

No theta or phi dependence because we expect the solution to depend on the distance from the source only

spherical polar with $R = |\vec{r}-\vec{r}'|$

$$\nabla^2 = \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

How to solve this in the Lorentz gauge

$$\frac{d^2G}{dR^2}+\frac{2}{R}\frac{dG}{dR}+\frac{\omega^2}{c^2}G = 0 \qquad (R\neq 0)$$

$$\frac{d^2}{dR^2}(GR)+\frac{\omega^2}{c^2}GR = 0$$

To fix the constant A, need to integrate both sides over a small sphere centered at R=0, with the delta fn in RHS

$$G = \frac{A}{R}e^{\pm i(\omega/c)R}$$

$$G \approx \frac{A}{R}\left(1 \pm i\frac{\omega}{c}R - \frac{1}{2}\frac{\omega^2}{c^2}R^2 \pm \ldots\right)$$

As $R\to 0$

$$G \approx \frac{A}{R}$$

$$\int_{\substack{small\\sphere}} d\tau\left(\nabla^2 G + \frac{\omega^2}{c^2}G\right) = -\int_{\substack{small\\sphere}} d\tau\,\delta(\vec{r}-\vec{r}\,')$$

$$-4\pi A \qquad 0 \qquad -1$$

The retarded/advanced potential

$$\tilde{V}(\vec{r},\omega) = \int d\tau' \left(\frac{\tilde{\rho}(\vec{r}',\omega)}{\epsilon_0}\right)\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}e^{\pm i(\omega/c)|\vec{r}-\vec{r}'|}\right]$$

$$V(\vec{r},t) = \int d\tau' \frac{d\omega}{2\pi}e^{-i\omega t}\left(\frac{\tilde{\rho}(\vec{r}',\omega)}{\epsilon_0}\right)\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}e^{\pm i(\omega/c)|\vec{r}-\vec{r}'|}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{1}{|\vec{r}-\vec{r}'|}\frac{d\omega}{2\pi}e^{-i\omega t}\left(\tilde{\rho}(\vec{r}',\omega)\right)\left[e^{\pm i(\omega/c)|\vec{r}-\vec{r}'|}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{1}{|\vec{r}-\vec{r}'|}\frac{d\omega}{2\pi}e^{-i\omega t}e^{\pm i(\omega/c)|\vec{r}-\vec{r}'|}\tilde{\rho}(\vec{r}',\omega)$$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{1}{|\vec{r}-\vec{r}'|}\rho\left(\vec{r}',t\pm\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

The retarded time arises naturally in Lorenz gauge solutions. It is not put in by some other considerations!

The retarded/advanced potential

The solution for A will have similar dependence on J, component by component.

The solution appears to say that the "information" about a change of charge at r' reaches the point r with speed c. This is an attractive physical interpretation – but works only for the potentials.

By the same logic, one might try to "retard" the solution for E and B and obtain the time dependent solution – IT DOESN'T WORK!

The actual E and B must be obtained by differentiating the potentials and they look very different. They may no longer fall off as 1/r^2

The "retarded" integral for the potentials is often non-trivial to do.

However everything about "radiation" is contained in that retarded potential term!

Radiation

The most important consequence of certain time varying charge and current configurations is radiation. A part of the E and B fields fall off as 1/r – a strikingly different behaviour.

These means that the Poynting vector integrated over a spherical surface may give a constant value as the r dependence of E x B and the surface area would cancel each other.

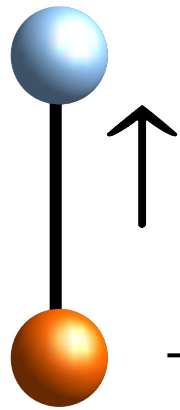
This outward energy flow is radiation from a source like a radio antenna or something else, like an accelerated charge.

The complete E and B fields created by an antenna/accelerating charge can be quite complicated. It is only one part that has the 1/r dependence. However this is the term which we would need to consider for calculating radiation.

The part of the field that falls off as 1/r is called the radiation field.

An oscillating (short) dipole

$z' = \frac{l}{2}$ $\quad q_0 \cos(\omega t)$



$I = -q_0 \omega \sin(\omega t)$

$z' = -\frac{l}{2}$ $\quad -q_0 \cos(\omega t)$

Overall electrically neutral

small size $\quad l \ll \frac{2\pi c}{\omega}$

far field $\quad l \ll |\vec{r}|$

$$V(\vec{r}, t) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos\omega\left(t - \frac{|\vec{r} - \hat{k}\, l/2|}{c}\right)}{|\vec{r} - \hat{k}\, l/2|} - \frac{\cos\omega\left(t - \frac{|\vec{r} + \hat{k}\, l/2|}{c}\right)}{|\vec{r} + \hat{k}\, l/2|}\right]$$

$$A_z(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} \frac{I(z', t - |\vec{r} - \hat{k} z'|/c)}{|\vec{r} - \hat{k} z'|}\, dz'$$

$$\approx \frac{\mu_0 I}{4\pi}\frac{l}{r}\sin\omega\left(t - \frac{|\vec{r}|}{c}\right) \qquad \text{where} \qquad I = -q_0\omega$$

No moving or accelerating charges in this...

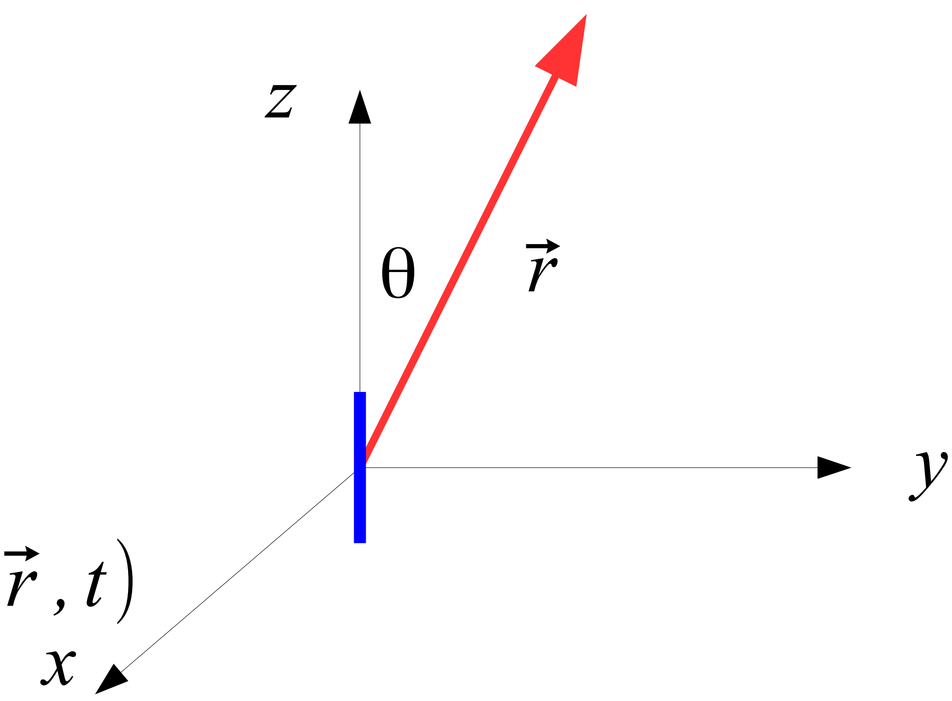
An oscillating (short) dipole

Question: Why did we not model the "oscillating dipole" as two charged balls on a spring ? This must give the same answer but will involve calculating the retarded potentials and fields due to moving/accelerating charges. We will do that later.

First we need to approximate the distances involved

$$|\vec{r} \pm \hat{k}\, l/2| = r\left(1 \pm \frac{l}{2r}\cos\theta\right)$$

$$\frac{1}{|\vec{r} \pm \hat{k}\, l/2|} = \frac{1}{r}\left(1 \mp \frac{l}{2r}\cos\theta\right)$$



Use these two to approximate $V(\vec{r}, t)$

The scalar potential with time variation

$$V(\vec{r},t) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos\omega\left(t-\frac{|\vec{r}-\hat{k}\,l/2|}{c}\right)}{|\vec{r}-\hat{k}\,l/2|} - \frac{\cos\omega\left(t-\frac{|\vec{r}+\hat{k}\,l/2|}{c}\right)}{|\vec{r}+\hat{k}\,l/2|}\right]$$

Use binomial and small angle approximation

$$\approx \frac{q_0\,l\cos\theta}{4\pi\epsilon_0 r}\left[\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)\right]$$

The first term $\left(\sim\frac{1}{r^2}\right)$ will reduce to electrostatic dipole as $\omega\to 0$

The second term $(\omega\neq 0)$ falls off slowly

This gives rise to the radiation term as $r\to\infty$

Is the gauge condition satisfied ?

Lorenz gauge : $\nabla . \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$$V(\vec{r},t) = \frac{q_0 l \cos\theta}{4\pi\epsilon_0 r}\left[\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$\frac{\partial V}{\partial t} = \frac{q_0 l \cos\theta}{4\pi\epsilon_0 r}\left[-\frac{1}{r}\omega \ \sin\omega\left(t-\frac{r}{c}\right) - \frac{\omega^2}{c}\cos\omega\left(t-\frac{r}{c}\right)\right]$$

$$A_z(\vec{r},t) = \frac{\mu_0 I}{4\pi}\frac{l}{r}\sin\omega\left(t-\frac{r}{c}\right)$$

where $I = -q_0\omega$

$$\frac{\partial A_z}{\partial z} = \frac{1}{c^2}\left(\frac{-q_0\omega l}{4\pi\epsilon_0}\right)\left[-\frac{1}{r^2}\frac{z}{r}\sin\omega\left(t-\frac{r}{c}\right)+\frac{\omega}{r}\cos\omega\left(t-\frac{r}{c}\right)\left(-\frac{z}{r}\right)\right]$$

Since $\frac{z}{r} = \cos\theta$, the two expressions are identical

We could have used this to calculate $V(\vec{r},t)$ from $A(\vec{r},t)$

Calculating $\vec{E}$ and $\vec{B}$ from V and $\vec{A}$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{\epsilon}}_r & r\,\hat{\boldsymbol{\epsilon}}_\theta & r\sin\theta\,\hat{\boldsymbol{\epsilon}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_z \cos\theta & -r A_z \sin\theta & 0 \end{vmatrix}$$

$$B_r = 0$$

$$B_\theta = 0$$

$$B_\phi = \frac{\mu_0 I}{4\pi}\frac{l}{r}\sin\theta\left[\frac{\omega}{c}\cos\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\sin\omega\left(t-\frac{r}{c}\right)\right]$$

There is one term which falls off as $\sim\frac{1}{r}$

$$B_\phi = \frac{\mu_0 I}{4\pi}\frac{l}{r}\sin\theta\left[\frac{\omega}{c}\cos\omega\left(t-\frac{r}{c}\right)\right]$$

Calculating $\vec{E}$ and $\vec{B}$ from V and $\vec{A}$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$E_r = \frac{q\,l\cos\theta}{4\pi\,\epsilon_0 r^2}\left[\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$E_\theta = \frac{q\,l\sin\theta}{4\pi\,\epsilon_0 r^2}\left[\left(\frac{1}{r}-\frac{\omega^2}{c^2}r\right)\cos\omega\left(t-\frac{r}{c}\right)-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$E_\phi = 0$$

Only component that falls off as $\sim\frac{1}{r}$:

$$E_\theta = -\frac{ql}{4\pi\,\epsilon_0}\frac{\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)$$

Power radiated by the dipole

Consider $\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$ over a sphere with $R\rightarrow\infty$

We only need to consider

$$E_\theta = -\frac{ql}{4\pi\epsilon_0}\frac{\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)$$

$$B_\phi = -\frac{\mu_0 ql}{4\pi}\frac{\omega^2}{c}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)$$

$$I_0=-q\omega$$

$$\oiint_R \vec{S}.d\vec{a} = \frac{R^2}{\mu_0}\int E_\theta B_\phi 2\pi\sin\theta d\theta$$

$$= \frac{q^2 l^2}{6\pi\epsilon_0 c^3}\omega^4\cos^2\omega\left(t-\frac{R}{c}\right)$$

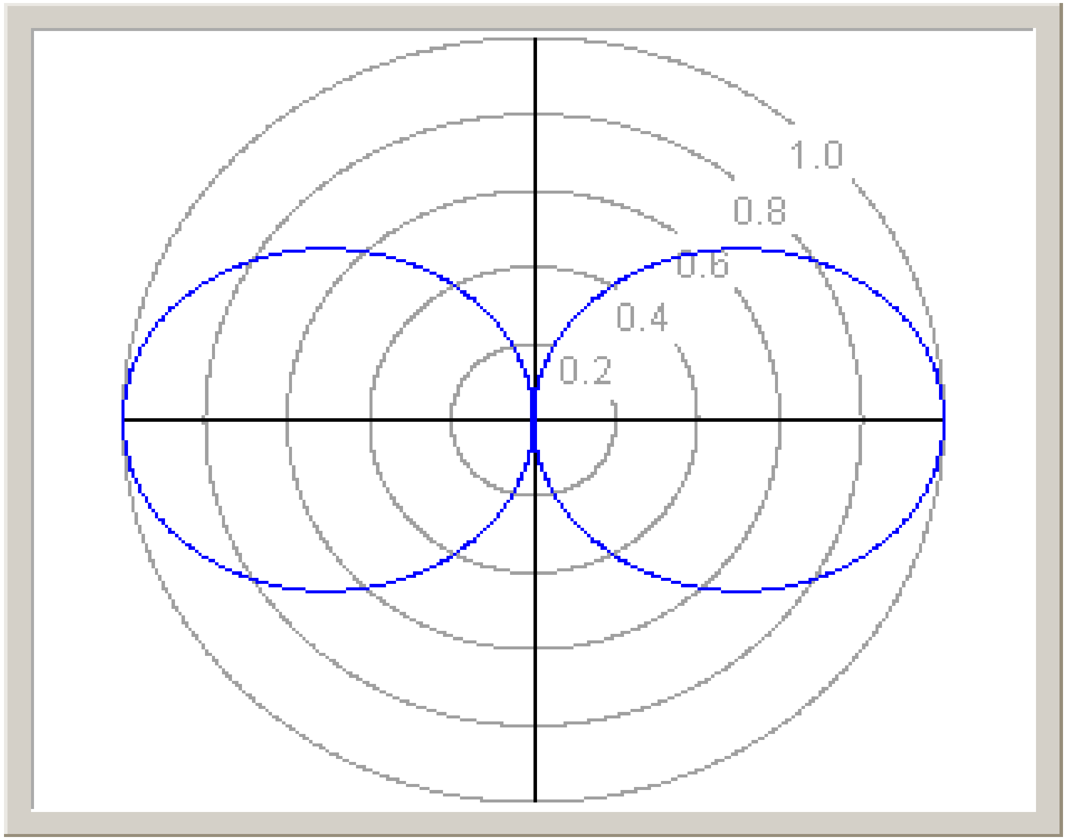
$$\langle P_{radiated}\rangle = \frac{1}{4\pi\epsilon_0}\frac{(ql)^2\omega^4}{3c^3} = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{l}{\lambda}\right)^2\frac{I_0^2}{2}$$

Both expressions are equivalent.

They show two different ways of viewing the source of radiation.

Either as a dipole or as a "current element" of an antenna.

Radiation pattern and antenna impedance



No intensity along the axis.

Maximum intensity on the equitorial plane.

In the polar plot the radial distance is the magnitude of the quantity at a certain angle.

The far field pattern tells us how much the dipole is radiating.

The near field pattern will be necessary if we want to calculate the effect of one dipole on another nearby dipole. (i.e. How would two antennas interfere, etc. ?)

Radiation pattern and antenna impedance

$$\langle P_{radiated} \rangle = \frac{1}{4\pi\epsilon_0}\frac{(ql)^2\omega^4}{3c^3} = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{l}{\lambda}\right)\frac{I_0^{\,2}}{2}$$

Radiated power = Real part of Impedance $\times$ r.m.s. current

The quantity $\sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega$ sets the impedance scale

This is called the Radiation resistance of an antenna

This does NOT tell us the reactive part of the impedenance.

Also the result is correct only for $l \ll \lambda$

$$\longleftarrow - - - - - - - \frac{2D^2}{\lambda} - - - - - - - \longrightarrow$$

Antenna dimension D

Reactive near field

$\approx 0.62\sqrt{\frac{D^3}{\lambda}}$

Radiative near field
Wavefront shape
keeps changing

Far field
(Fraunhoffer)

The half wave dipole

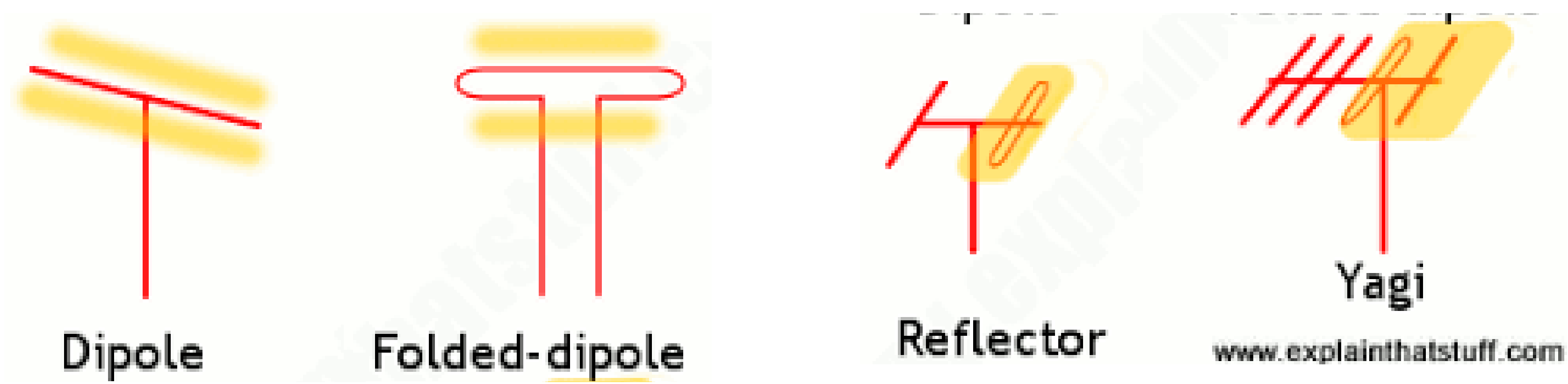
The short dipole result will not hold unless $l \ll \lambda$.

$l = \lambda/2$ is a common configuration called a half wave antenna.

But setting $\dfrac{l}{\lambda} = \dfrac{1}{2}$ in the earlier formula won't work!

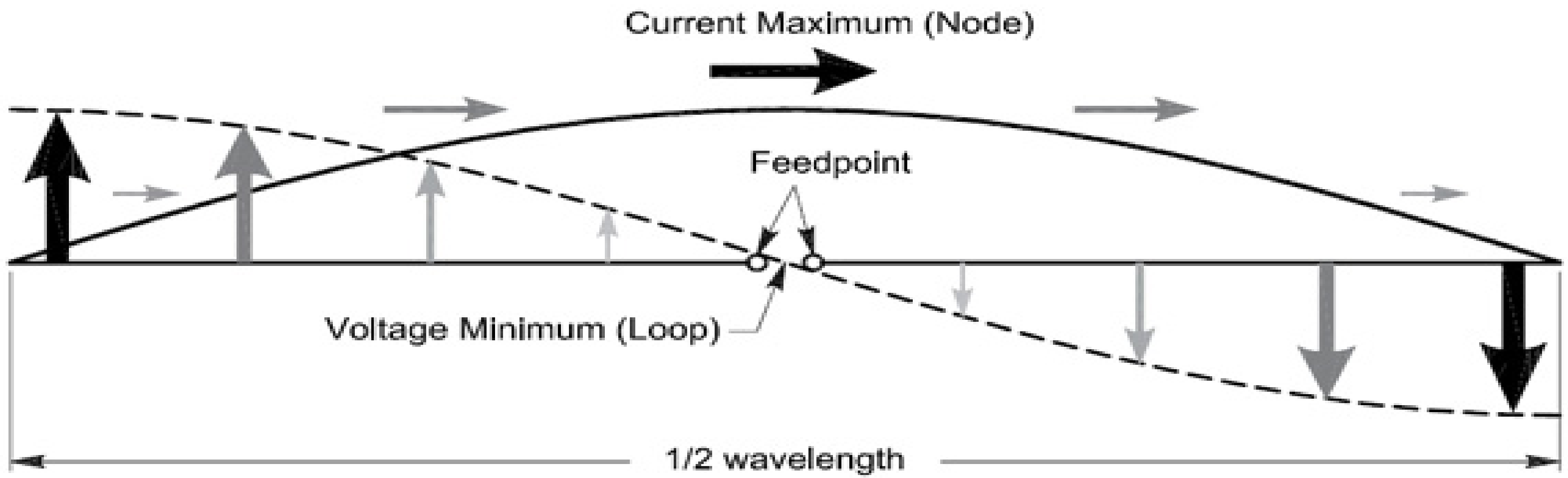
Also : If $l = \lambda$ IT WILL NOT RADIATE AT ALL! Why?

For $l > \lambda/2$ some parts will start having oppposite currents....



Variants of the dipole antenna.

The half wave dipole



The "short" dipole that we analyzed can be used to build up a solution, if we know the current at each point of the dipole.

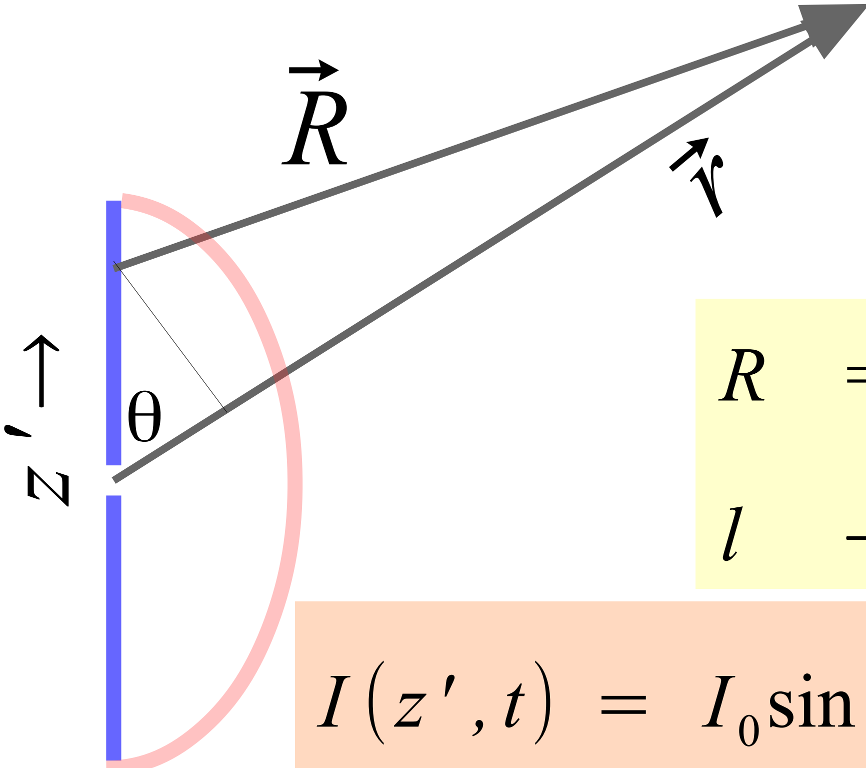
However the current at each point must be consistent with the "near field" produced by the other parts.

This makes the "exact" solution a difficult self-consistent problem.

We generally assume a reasonable current pattern that goes to zero at the ends and is maximum at the feed-point.

It so happens that the "numerically exact" solution agree quite closely with the result from the profile shown.

The half wave dipole



small dipole result derived earlier

$$E_\theta = -\frac{ql}{4\pi\epsilon_0}\frac{\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)$$

$$R = r - z'\cos\theta \qquad \left(-\frac{\lambda}{4} < z' < \frac{\lambda}{4}\right)$$

$$l \to dz' \qquad I_0 = -q\omega$$

$$I(z',t) = I_0 \sin\omega t \cos\left(\frac{2\pi z'}{\lambda}\right)$$

Far field due to a segment between z' to $z'+dz'$

$$dE_\theta = \left(\frac{I_0}{4\pi\epsilon_0 c}\right)\frac{\sin\theta}{R}\frac{\omega}{c}\cos\omega\left(t-\frac{R}{c}\right)\cos\left(\frac{2\pi z'}{\lambda}\right)dz'$$

$$dB_\phi = \left(\frac{\mu_0 I_0}{4\pi}\right)\frac{\sin\theta}{R}\frac{\omega}{c}\cos\omega\left(t-\frac{R}{c}\right)\cos\left(\frac{2\pi z'}{\lambda}\right)dz'$$

The half wave dipole

$$u = \frac{2\pi z'}{\lambda} \qquad R = r - z'\cos\theta \qquad \text{(change variables)}$$

$$K = \int_{-\pi/2}^{\pi/2} \frac{1}{R}\cos\omega\left(t - \frac{R}{c}\right)\cos u\, du$$

$$\approx \frac{1}{r}\int_{-\pi/2}^{\pi/2} \cos\left[\omega\left(t - \frac{r}{c}\right) + u\cos\theta\right]\cos u\, du \qquad (r \gg z')$$

$$= \frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right)\int_{-\pi/2}^{\pi/2} \cos(u\cos\theta)\cos u\, du \;+$$

$$\frac{1}{r}\sin\omega\left(t - \frac{r}{c}\right)\int_{-\pi/2}^{\pi/2} \sin(u\cos\theta)\cos u\, du \qquad = 0$$

$$K = \frac{2}{r}\cos\omega\left(t - \frac{r}{c}\right)\frac{\cos(\pi/2\cos\theta)}{\sin^2\theta}$$

The half wave dipole

$$E_\theta = \left(\frac{I_0}{4\pi\epsilon c}\right)K = \left(\frac{I_0}{2\pi\epsilon c r}\right)\cos\omega\left(t-\frac{r}{c}\right)\frac{\cos(\pi/2\cos\theta)}{\sin\theta}$$

$$B_\phi = \left(\frac{\mu_0 I_0}{4\pi}\right)K = \left(\frac{\mu_0 I_0}{2\pi r}\right)\cos\omega\left(t-\frac{r}{c}\right)\frac{\cos(\pi/2\cos\theta)}{\sin\theta}$$

Integrating the Poynting vector over a large sphere

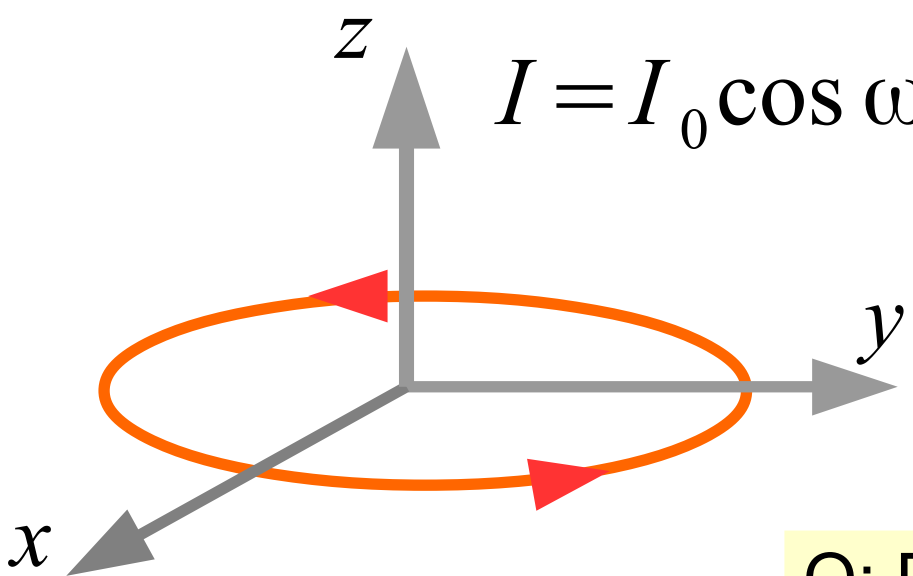
$$\langle P_{radiated}\rangle = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}I_0^{\,2}\int_0^\pi\left(\frac{\cos(\pi/2\cos\theta)}{\sin\theta}\right)^2\sin\theta\, d\theta$$

$$= 73\,(\text{ohms})\times\left(\frac{I_0^{\,2}}{2}\right)$$

This approximately 75 Ohms impedance is often encountered in dealing with cables connecting antennas to amplifiers etc. What is the reason ? [Discuss later]

Magnetic dipole radiation

We saw that an oscillating electric dipole radiates. A natural question is what does an oscillating magnetic dipole do ?



$$I = I_0 \cos \omega t$$

Such a current will produce no electric potential. Why ?

The loop area → 0
m = I x area = constant

Q: Does it satisfy the Lorenz gauge?
Does it give the correct static limit ?

The solution

No Radiation   Radiation

$$V(\vec{r}, t) = 0$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 m}{4\pi}\left(\frac{\sin\theta}{r}\right)\left[\underline{\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)} - \underline{\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)}\right]\hat{\epsilon}_\phi$$

Magnetic dipole radiation

The far field

$$\vec{E} = \frac{-\partial \vec{A}}{\partial t} = \frac{\mu_0 m \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\epsilon}_\phi$$

$$\vec{B} = \nabla\times\vec{A} = \frac{-\mu_0 m \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\epsilon}_\theta$$

The radiated power $\left\langle P_{radiated}\right\rangle = \frac{1}{4\pi\epsilon_0}\frac{m^2\omega^4}{3c^5}$

This power is small in comparison to an electric dipole of similar size with $I_0 \rightarrow q\omega$ and $\pi a \rightarrow d$ where a is the radius of the current loop and d the dipole length

Potential due to a *moving* point charge

Since

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{1}{|\vec{r}-\vec{r}\,'|}\rho\left(\vec{r}\,',t\pm\frac{|\vec{r}-\vec{r}\,'|}{c}\right)$$

One might think that for a point charge

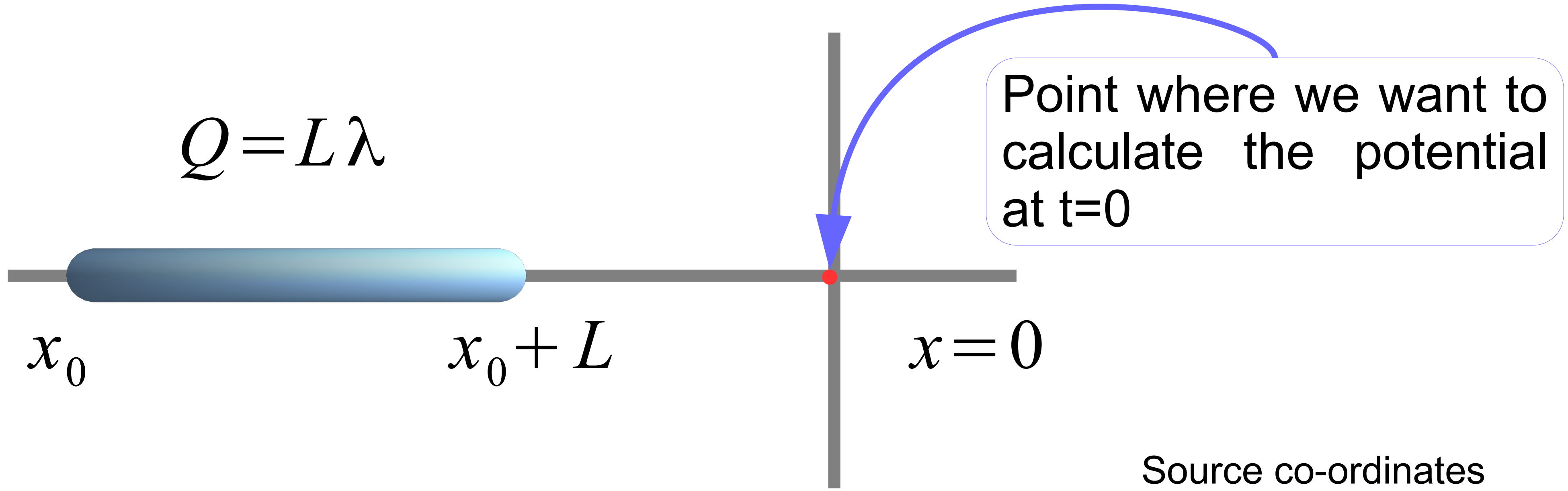
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}\,'_{ret}|}$$

where $\vec{r}\,'_{ret}$ is the vector to the retarded position

THIS HOWEVER IS WRONG !!

Why it is wrong and what the correct form is was figured out around 1901-02 only a couple of years before the special theory of relativity was published! The reason is quite non-trivial.....

A line charge moving along x axis



Source co-ordinates are primed.

$$
\begin{aligned}
\rho(x') &= \lambda \Theta(x' - x_0') \times \Theta(x_0' + L - x') \\
x_0' &= A + ut \\
\rho(x', t) &= \lambda \Theta(x' - A - ut) \times \Theta(A + L + ut - x') \\
t_r &= t - \frac{|x'|}{c}
\end{aligned}
$$

A line charge moving along x axis

$$\rho(x',t_r) = \lambda\Theta\left[x'-A-u.\left(t-\frac{|x'|}{c}\right)\right]\times\Theta\left[A+L+u.\left(t-\frac{|x'|}{c}\right)-x'\right]$$

$$= \lambda\Theta\left[x'\left(1-\frac{u}{c}\right)-A-ut\right]\times\Theta\left[A+L+ut-x'\left(1-\frac{u}{c}\right)\right]$$

In our drawing x' < 0 so the sign of x' and its absolute value |x'| will be opposite

The function is non-zero if both the following are met

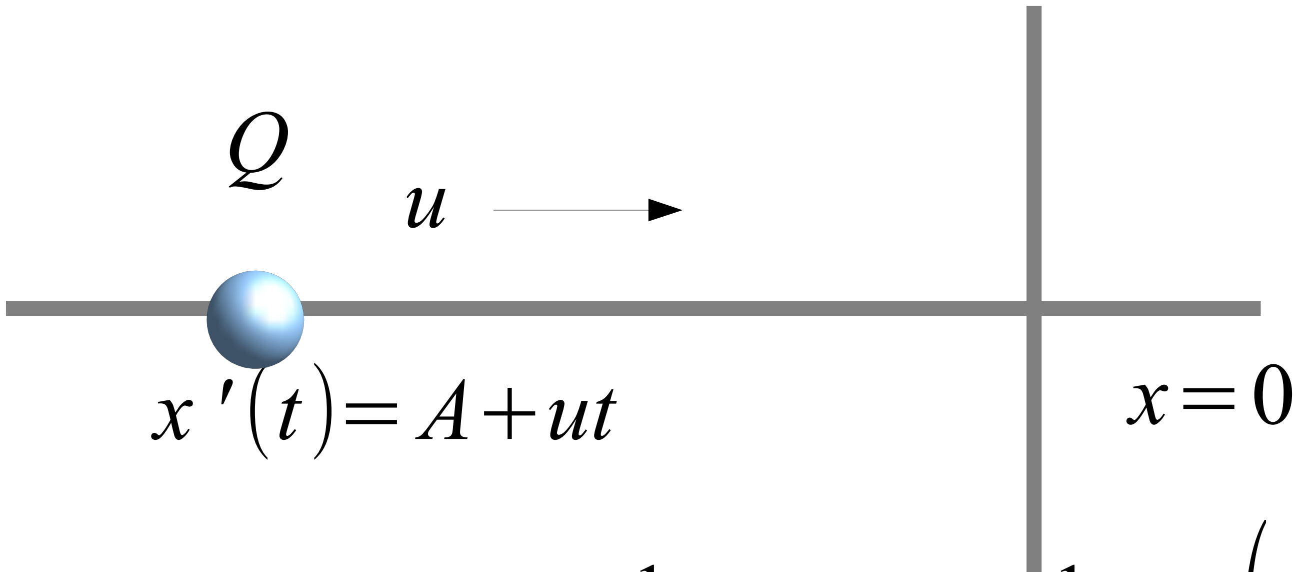
$$\left.\begin{aligned} x' &> \frac{A+ut}{1-u/c} \\ x' &< \frac{A+L+ut}{1-u/c} \end{aligned}\right\}$$

The length over which it is non-zero is $\dfrac{L}{1-u/c}$ NOT L

No change in the linear density λ : Also correct as $L \to 0$

Note : The factor u/c has NO connection with special relativity

Point charge : $L \to 0$ limit of a line charge



$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{1}{|\vec{r} - \vec{r}\,'|} \rho\left(\vec{r}\,', t \pm \frac{|\vec{r} - \vec{r}\,'|}{c}\right)$$

$$V(x=0, t=0) = \frac{1}{4\pi\epsilon_0} \int \underset{\cdots}{dx'} \frac{1}{|\vec{0} - \vec{x}\,'|} \rho\left(x', 0 - \frac{|x'|}{c}\right)$$

$$\rho \neq 0 \quad \text{only for} \quad \boxed{\frac{A+ut}{1-u/c}} < x' < \boxed{\frac{A+L+ut}{1-u/c}}$$

$$= \frac{1}{4\pi\epsilon_0} \left(\frac{L}{1-u/c}\right) \frac{1-u/c}{A} \frac{Q}{L} = \frac{1}{4\pi\epsilon_0} \frac{Q}{A}$$

The retarded position

$$V(0,0) = \frac{1}{4\pi\epsilon_0}\frac{Q}{A}$$ is not very useful

The constant A is arbitrary and shouldn't be there....

$$\left.\begin{aligned} x'(t_r) &= A + ut_r \\ t_r &= \frac{x'(t_r)}{c} \end{aligned}\right\} \Rightarrow x'(t_r) = \frac{A}{1-u/c}$$

So the expression for $V(0,t)$ can be written as

$$V(0,t) = \frac{1}{4\pi\epsilon_0}\left(\frac{L}{1-u/c}\right)\frac{1-u/c}{A}\frac{Q}{L} \Rightarrow \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{1-u/c}\right)\frac{1}{x'(t_r)}$$

Notice that we have got an expression for the potential of a moving point charge for a very restricted situation. We now need to generalise this for a charge moving in any given trajectory.

Point charge in arbitrary motion

The generalisation can be done in mutliple ways. One way is to emphasize the origin of the (1-u/c) factor as resulting from an apparent change in the volume over which the source coordinate integration has a non-zero integrand.

Another way is to hide that by using a delta-function trick. We will see both.

The resulting expressions are called the Lienard-Wiechart potentials – one of the most remarkable results of classical electromagnetism (these were derived about 5 years before the special theory of relativity).

We will see that the results we get (though it is quite long drawn) are exactly the same that Lorentz transformation to a moving frame would give.

Problem : Given $\vec{r}$ and t how to find $t_r = t - \Delta t$?

Equation of the trajectory $\vec{r}'(t)$ must be known

$\Rightarrow$ at $t - \Delta t$ the particle was at $\vec{r}'(t - \Delta t)$

$$\Rightarrow \boxed{|\vec{r} - \vec{r}'(t - \Delta t)|} = \boxed{c \Delta t}$$

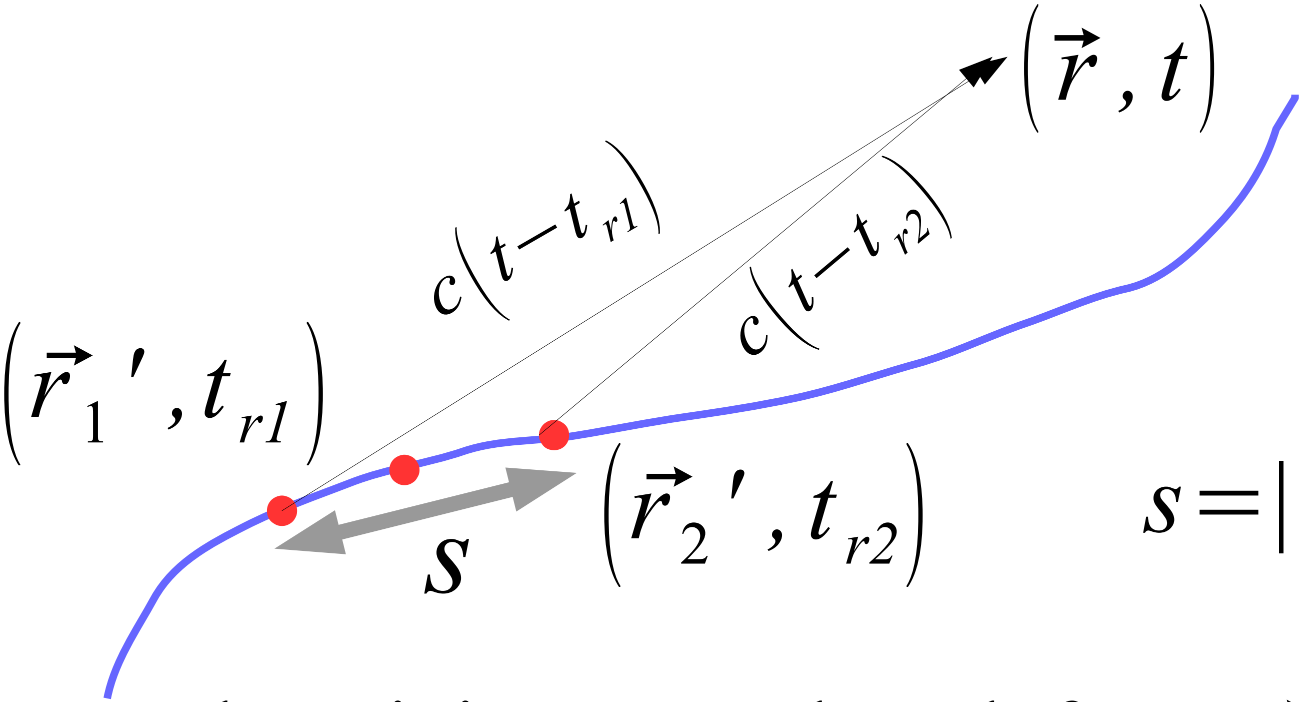
$$\equiv |\vec{r} - \vec{r}'(t_r)| = c(t - t_r)$$

The length from the point of observation to the retarded position must have been "traversed by light" (but this is not real light!!) in the time interval (current time – retarded time)

The equation gives the retarded time implicitly. Usually the algebraic equation involves squaring both sides.. often making it a messy quadratic to solve !

Solve for retarded time → find retarded position → calculate the position vector from observation point to retarded position.

For a fixed path only one retarded position is possible...



$$s = |\vec{r}_1' - \vec{r}_2'|$$

$s =$ The minimum arc length from $\vec{r}_1'$ to $\vec{r}_2'$

$$s + c(t - t_{r1}) > c(t - t_{r2}) \qquad \Delta t_r = |t_{r2} - t_{r1}|$$

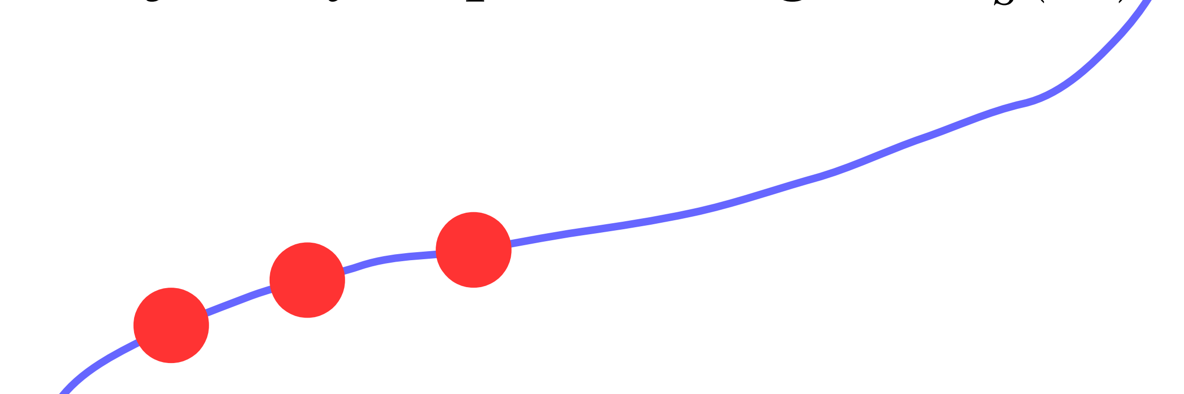
two sides of a triangle must be greater than the third side

$$\Rightarrow s = |v_{av}| \Delta t_r > c \Delta t_r \Rightarrow |v_{av}| > c$$

Having two retarded points on the trajectory is not possible.
It would require the particle to move faster than *c*

Another way of doing this.....

Trajectory of point charge : $\vec{r}_S(t')$



In this case we assume that it is a point charge, right from the beginning.

$$\rho(\vec{r}\,', t_r') = q\,\delta(\vec{r}\,' - \vec{r}_S(t_r')) = q\int dt'\,\delta^3(\vec{r}\,' - \vec{r}_S(t')) \times \delta(t' - t_r')$$

$$\boxed{V(\vec{r}, t) \quad = \quad \frac{1}{4\pi\epsilon_0}\int d\tau'\,\frac{1}{|\vec{r} - \vec{r}\,'|}\,\rho\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}$$

$$= \quad \frac{1}{4\pi\epsilon_0}\int d\tau'\,dt'\,\frac{q\,\delta(\vec{r}\,' - \vec{r}_S(t'))}{|\vec{r} - \vec{r}\,'|}\,\delta(t' - t_r')$$

$$= \quad \frac{q}{4\pi\epsilon_0}\int dt'\,\frac{1}{|\vec{r} - \vec{r}_S(t')|}\,\delta(t' - t'_r)$$

Another way of doing this.....

The meaning of the integral has to be understood clearly.

We chose a t' first → for a choice of t' the trajectory gives one position r(t') → For this position calculate the retarded time that appears in the delta function. The quantity that appears in the argument of the delta function is itself a function of t'.

$$\delta(t'-t_r') = \delta\left(t'-\left(t-\frac{|\vec{r}-\vec{r}\,'(t')|}{c}\right)\right)=\delta(f(t'))$$

Now use the fact that $\delta(f(x))=\sum \frac{\delta(x-x_i)}{|f'(x)|}$

The sum runs over all zeros of $f(x)$

But there is only one "point" that can contribute to the integral at the end (we just proved it earlier). But that point will still "stretch out" due to the motion of the charge

Another way of doing this.....

$$\delta(t'-t'_r)=\frac{\delta(t'-t_r)}{\frac{\partial}{\partial t'}(t'-t_r')} \qquad \text{where} \quad t_r'=t-\frac{|\vec{r}-\vec{r}'|}{c}$$

$$\frac{\partial}{\partial t'}(t'-t_r') = 1+\frac{1}{c}\frac{1}{|\vec{r}-\vec{r}'|}\left[-(x-x')\frac{\partial x'}{\partial t'}-(y-y')\frac{\partial y'}{\partial t'}-(z-z')\frac{\partial z'}{\partial t'}\right]$$

$$= 1-\frac{\vec{v}'}{c}.\hat{R}_r \qquad \text{where} \quad \hat{R}_r=\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

Here $\hat{R}_r$ and $\vec{v}'$ must be evaluated at the retarded time t_r

$$\left.\begin{array}{l} \left|\frac{\vec{v}'}{c}\right| < 1 \\ \hat{R}_r \text{ is a unit vector} \end{array}\right\} \text{we can skip the overall modulus sign}$$

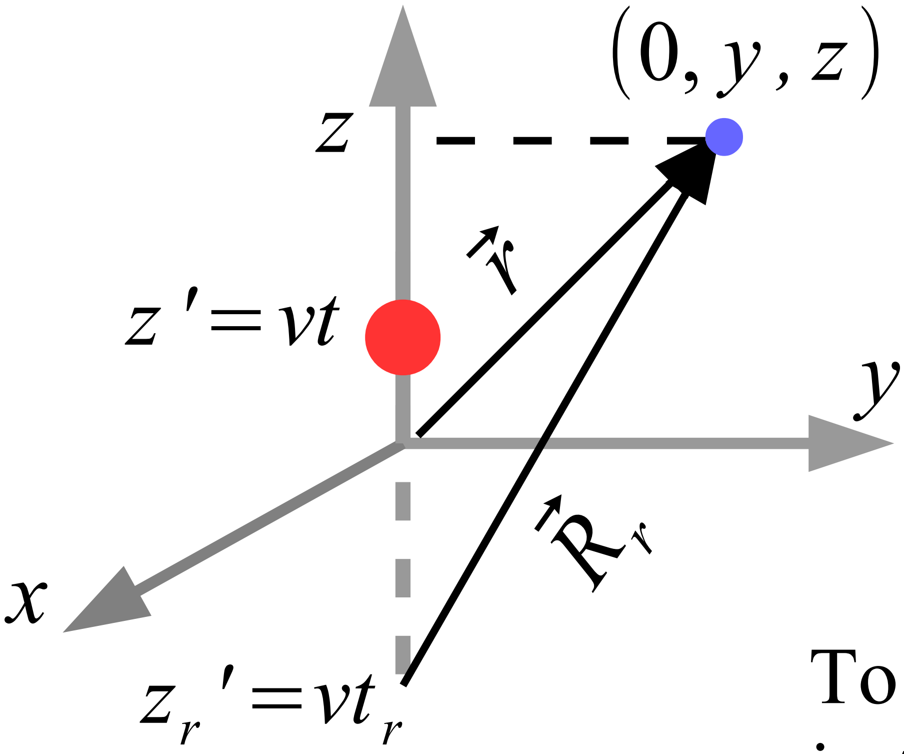
The Lienard-Wiechart potential

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{1-\vec{\beta}.\hat{R}_r}\right)\frac{1}{R_r}$$

$$\vec{A}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{\beta}}{c}\right)\left(\frac{1}{1-\vec{\beta}.\hat{R}_r}\right)\frac{1}{R_r}$$

$$= \frac{\mu_0}{4\pi}(q\vec{v})\left(\frac{1}{1-\vec{\beta}.\hat{R}_r}\right)\frac{1}{R_r}$$

Point charge in uniform motion along z axis



$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(1 - \vec{\beta} . \hat{R}_r\right) R_r}$$

R_r is the vector from the retarded position to point of observation

$\hat{R}_r$ is the unit vector along $\vec{R}_r$

To determine the denominator in terms of y, z, t variables only

$$R_r = c(t - t_r)$$

$$\frac{\vec{v}}{c} . \vec{R}_r = \frac{v}{c}(z - z_r') = \beta(z - vt_r)$$

$$c^2 (t - t_r)^2 = (z - vt_r)^2 + y^2$$

Strategy: determine the retarded time using the third eqn → Then use that in the first two equations → Subtract second eqn from the first eqn

Point charge in uniform motion along z axis

$$ct_r = \frac{(ct-\beta z)-\sqrt{(z-vt)^2+y^2(1-\beta^2)}}{1-\beta^2}$$

Why have we picked the -ve sign only ?

$$R_r = c(t-t_r) = \frac{\beta(z-vt)+\sqrt{(z-vt)^2+y^2(1-\beta^2)}}{1-\beta^2}$$

$$z_r' = vt_r$$

$$\beta(z-z_r') = \frac{\beta(z-vt)+\beta^2\sqrt{(z-vt)^2+y^2(1-\beta^2)}}{1-\beta^2}$$

$$\left(1-\frac{\vec{v}\cdot\hat{R}_r}{c}\right)R_r = \sqrt{(z-vt)^2+y^2(1-\beta^2)}$$

For $t=0$, we must get $t_r<0$
OR take $\beta\rightarrow 0$ then
match with the expected result

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(z-vt)^2+y^2(1-\beta^2)}}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

Uniformly moving point charge : $\vec{E}$ and $\vec{B}$

Using the rotational symmetry about z axis
The expressions can be easily generalised to

$$\begin{aligned} V(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(z-vt)^2+(x^2+y^2)(1-\beta^2)}} \\ \vec{A}(\vec{r},t) &= \frac{\vec{v}}{c^2}V(\vec{r},t) \end{aligned}$$

We now need to calculate the fields

$$\vec{E}=-\nabla V-\frac{\partial\vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B}=\nabla\times A$$

Since $\vec{A}$ and $\vec{v}$ (constant) point in the same direction

$$\vec{B}=\nabla\times\left(\frac{\vec{v}}{c^2}V(\vec{r},t)\right)=-\frac{\vec{v}}{c^2}\times\nabla V=\frac{\vec{v}}{c^2}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=\frac{\vec{v}}{c^2}\times\vec{E}$$

Uniformly moving point charge : $\vec{E}$ and $\vec{B}$

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(z-vt)^2+(x^2+y^2)(1-\beta^2)}} \qquad \beta=\frac{v}{c}$$

$$\frac{\partial V}{\partial x} = -\frac{q}{4\pi\epsilon_0}\frac{x(1-\beta^2)}{\left((z-vt)^2+(x^2+y^2)(1-\beta^2)\right)^{3/2}}$$

$$\frac{\partial V}{\partial y} = -\frac{q}{4\pi\epsilon_0}\frac{y(1-\beta^2)}{\left((z-vt)^2+(x^2+y^2)(1-\beta^2)\right)^{3/2}}$$

$$\frac{\partial V}{\partial z} = -\frac{q}{4\pi\epsilon_0}\frac{(z-vt)}{\left((z-vt)^2+(x^2+y^2)(1-\beta^2)\right)^{3/2}}$$

Since $\vec{A} \parallel \vec{v}$ only A_z exists

$$\frac{\partial A_z}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{\beta^2(z-vt)}{\left((z-vt)^2+(x^2+y^2)(1-\beta^2)\right)^{3/2}}$$

Uniformly moving point charge : $\vec{E}$ and $\vec{B}$

$$E_x = -\frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0}\frac{x\left(1-\beta^2\right)}{\left(\left(z-vt\right)^2+\left(x^2+y^2\right)\left(1-\beta^2\right)\right)^{3/2}}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{q}{4\pi\epsilon_0}\frac{y\left(1-\beta^2\right)}{\left(\left(z-vt\right)^2+\left(x^2+y^2\right)\left(1-\beta^2\right)\right)^{3/2}}$$

$$E_z = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{\left(1-\beta^2\right)\left(z-vt\right)}{\left(\left(z-vt\right)^2+\left(x^2+y^2\right)\left(1-\beta^2\right)\right)^{3/2}}$$

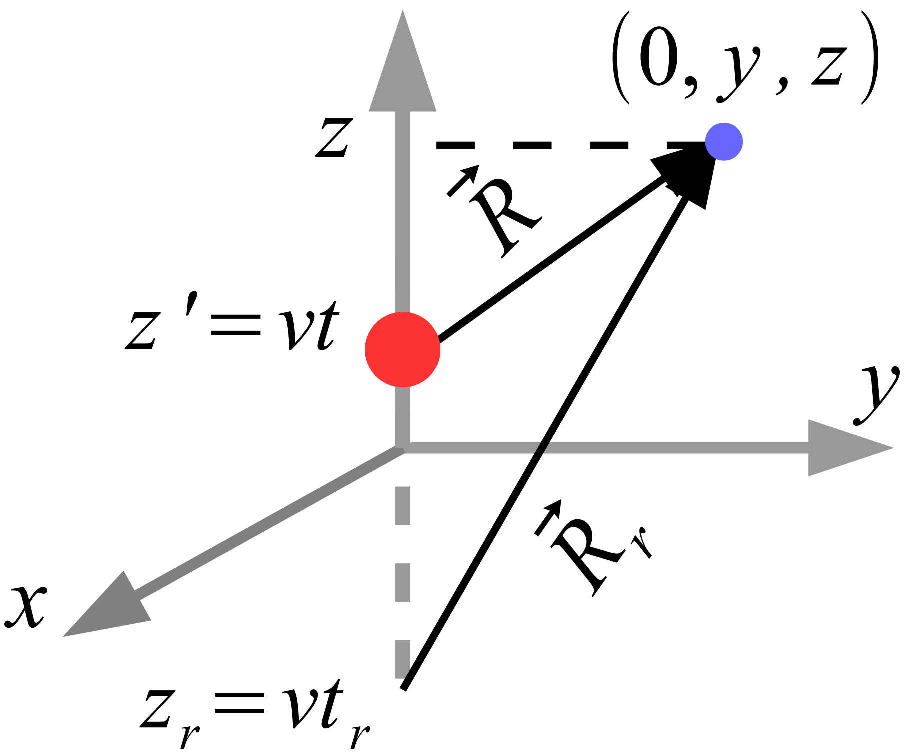
$$B_x = -\beta\frac{E_y}{c}$$

$$B_y = \beta\frac{E_x}{c}$$

$$B_z = 0$$

t is the current time, NOT retarded time
The inverse square nature of E is preserved.
B revolves round z axis as expected.

Uniformly moving point charge : $\vec{E}$ and $\vec{B}$



$\vec{R}$ $\rightarrow$ connects the observer to the CURRENT position of the charge

θ $\rightarrow$ is the angle between $\vec{R}$ and $\vec{v}$

$$\vec{R} = x\hat{i} + y\hat{j} + (z - vt)\hat{k}$$

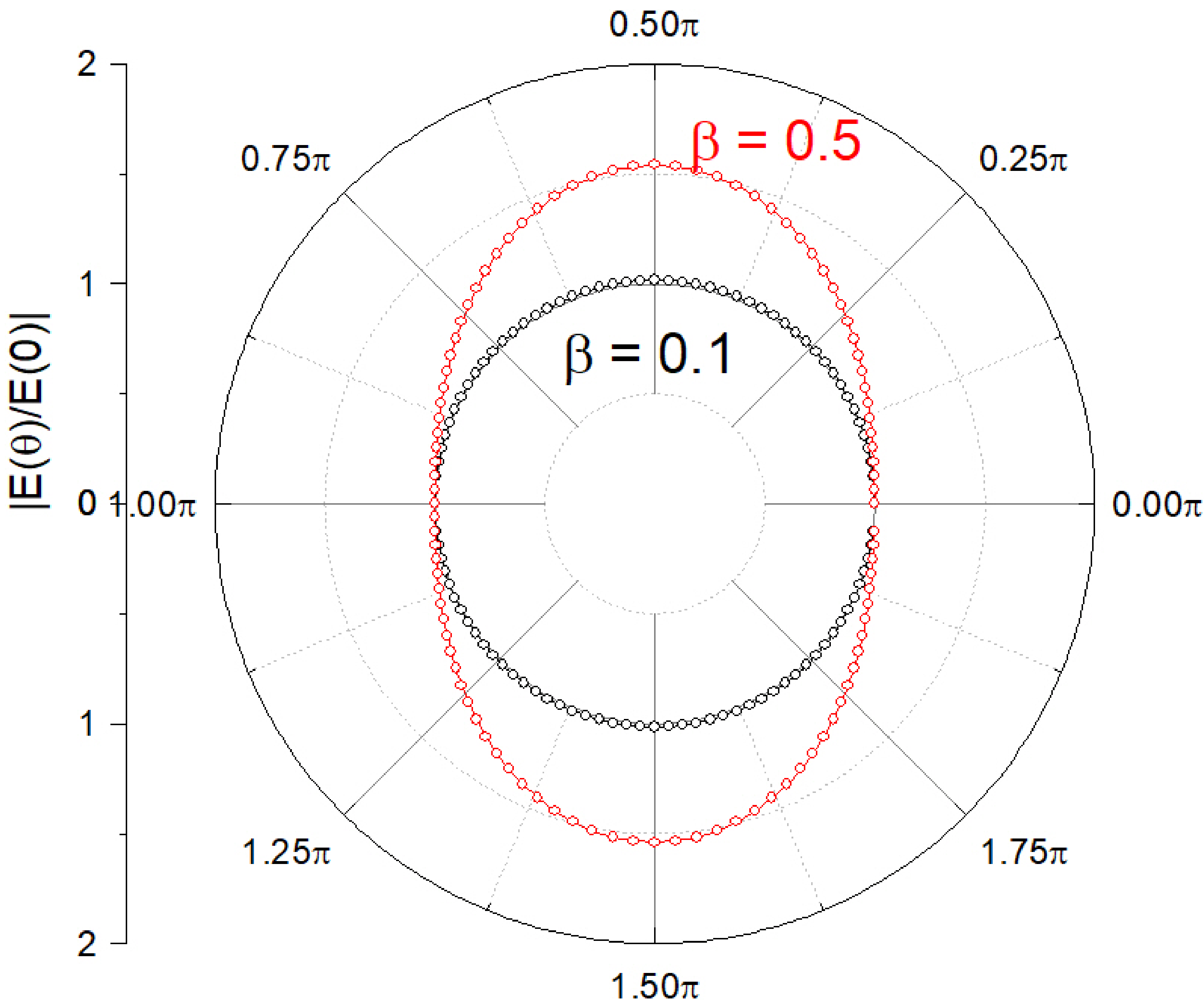
Another way of writing the result

$$\vec{E} = \frac{q}{4\pi\epsilon_0}(1-\beta^2)\frac{\vec{R}}{R^3}\frac{1}{(1-\beta^2 \sin^2\theta)^{3/2}}$$

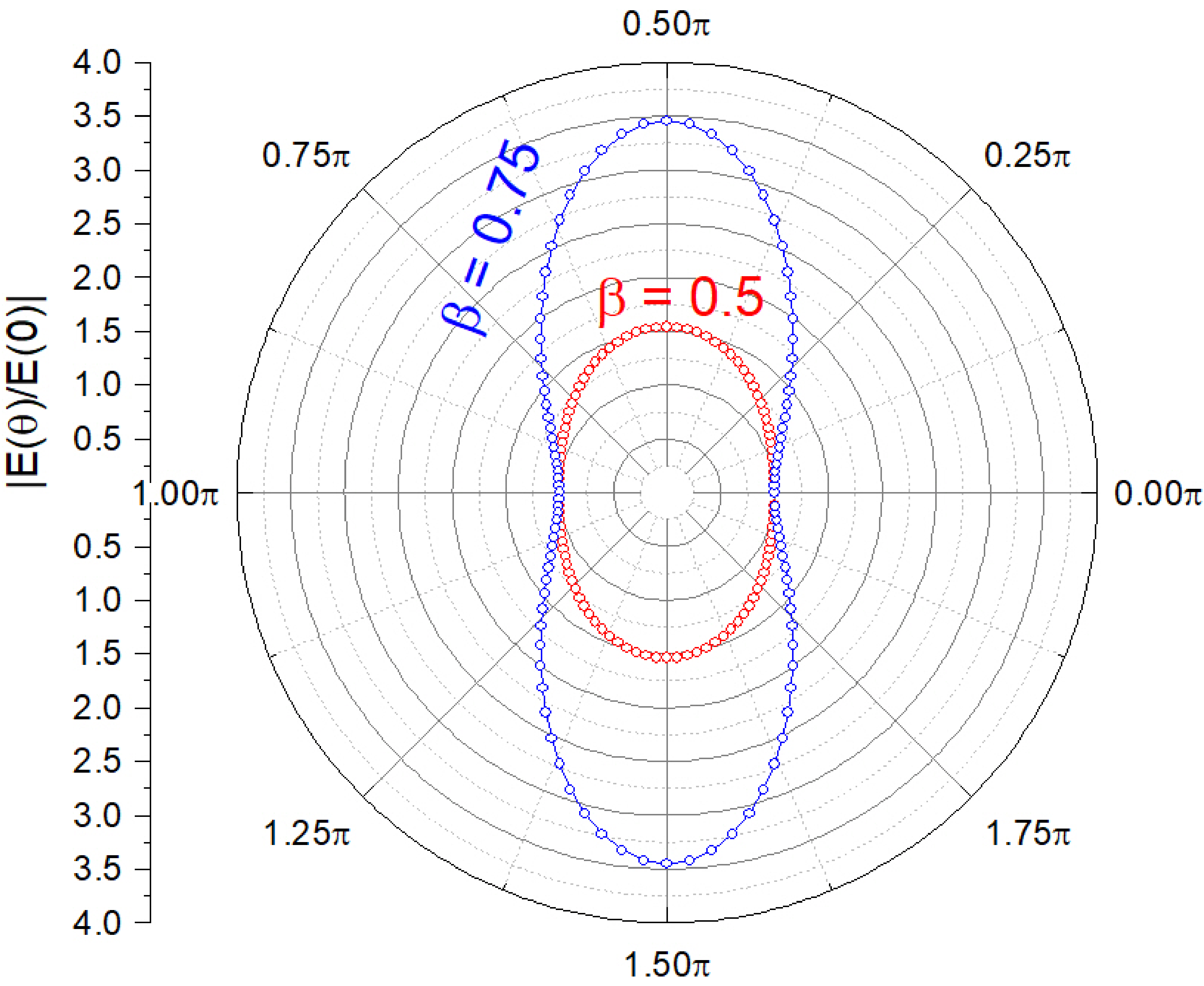
$\vec{E}$ remains radial

but is weakened in forward and backward directions

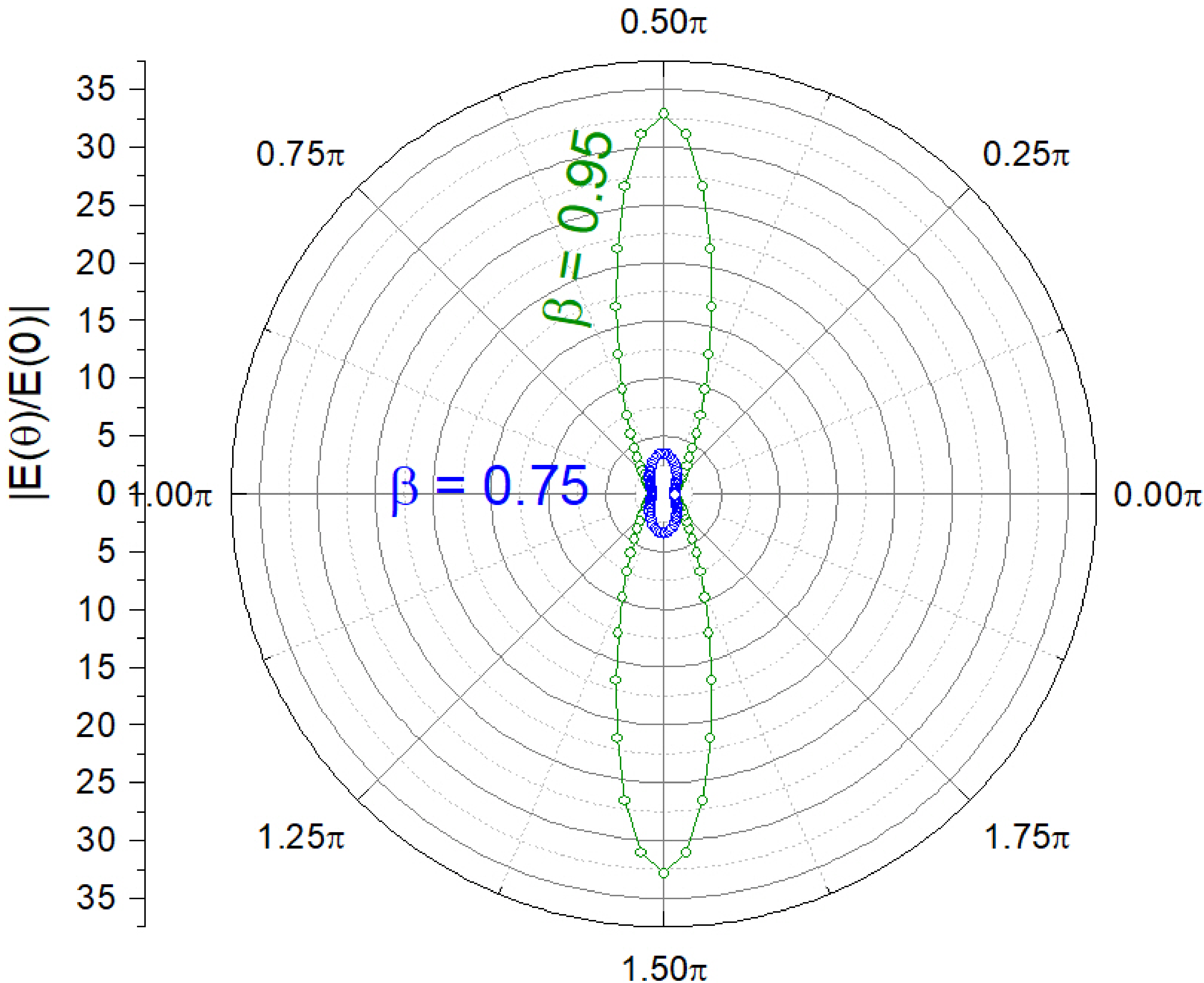
The forward and transverse directions



The forward and transverse directions



The forward and transverse directions



The consistency with special relativity

If we observe the charge from its rest frame, then E must be the Coulomb field and B must be zero.

Suppose we go to another intertial frame moving with velocity v. How would the E and B in these two frames (due to the same point charge) be connected? We should be able to apply Lorentz transformation to E = 1/r^2 and B =0 fields and obtain the answer.

The result obtained from Lorentz transformation agrees exactly with the results we deduced from the Lienard-Wiechart potentials.

This is remarkable – it works because special relativity is "built-in" in Maxwell's equations. The results we obtain will always be consistent with special relativity.

Accelerated point charge : $\vec{E}$ and $\vec{B}$

$\vec{E}$ and $\vec{B}$ fields of an accelerated point charge is one of the key problems of electrodynamics.

MESSY problem :EIGHT variables and their derivatives!

We evaluate the field at (x, y, z, t) is the point $[\vec{r} = (x, y, z)]$

(x', y', z', t_r) is the retarded position and retarded time

$\vec{R}_r = \vec{r} - \vec{r}\,'$ is another notation we will use

$$c(t - t_r) = R_r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

NOTE : The relevant velocity of the charge is :

$\vec{v} = \dfrac{\partial \vec{r}\,'}{\partial t_r}$ and not $\dfrac{\partial \vec{r}\,'}{\partial t}$

Framing the problem

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{1-\frac{\vec{v}}{c}.\hat{R}_r}\right)\frac{1}{R_r} = \frac{qc}{4\pi\epsilon_0}\left(\frac{1}{c-\vec{v}.\hat{R}_r}\right)\frac{1}{R_r}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

Expression for V will have x,y,z,t, and x',y',z' and t_r

We need derivatives w.r.t. x, y, z, t: $\begin{cases} \vec{E} = \nabla V - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times A \end{cases}$

THINK: In general we CANNOT eliminate (x', y', z', t)

We will frequently encounter derivatives of the retarded velocity & position w.r.t the current time and position. How do we do these ? We need to sort these out first!

Derivatives of the retarded variables

$$c(t-t_r) = R_r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$-c\frac{\partial t_r}{\partial x} = \frac{\partial R_r}{\partial x} \quad \Rightarrow \quad -c\nabla t_r = \nabla R_r$$

$$\frac{\partial R_r}{\partial x} = \frac{1}{R_r}\left[(x-x')\left(1-\frac{\partial x'}{\partial t_r}\frac{\partial t_r}{\partial x}\right)+(y-y')\left(-\frac{\partial y'}{\partial t_r}\frac{\partial t_r}{\partial x}\right)+ (z-z')\left(-\frac{\partial z'}{\partial t_r}\frac{\partial t_r}{\partial x}\right)\right]$$

$$= \frac{1}{R_r}\left[(x-x')\left(1-v_x\frac{\partial t_r}{\partial x}\right)+(y-y')\left(-v_y\frac{\partial t_r}{\partial x}\right)+(z-z')\left(-v_z\frac{\partial t_r}{\partial x}\right)\right]$$

$$-c\frac{\partial t_r}{\partial x} = \frac{(x-x')}{R_r} - \frac{\vec{v}.\vec{R}_r}{R_r}\frac{\partial t_r}{\partial x}$$

$$\boxed{\nabla t_r = -\frac{1}{c}\nabla R_r = -\frac{\vec{R}_r}{cR_r-\vec{v}.\vec{R}_r} = -\frac{1}{c}\frac{\hat{R}_r}{1-\frac{\vec{v}}{c}.\hat{R}_r} \quad \cdots\cdots\cdots DR(1)}$$

Derivatives of the retarded variables

$$c\left(t - t_r\right) = R_r = \sqrt{\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2}$$

$$c\left(1 - \frac{\partial t_r}{\partial t}\right) = \frac{1}{R_r}\left[-\left(x - x'\right)\frac{\partial x'}{\partial t_r} - \left(y - y'\right)\frac{\partial y'}{\partial t_r} - \left(z - z'\right)\frac{\partial z'}{\partial t_r}\right]\frac{\partial t_r}{\partial t}$$

$$\frac{\partial t_r}{\partial t} = \frac{cR_r}{cR_r - \vec{v}\,.\,\vec{R}_r} = \frac{1}{1 - \frac{\vec{v}}{c}\,.\,\hat{R}_r} \quad \cdots\cdots DR(2)$$

$$\frac{\partial R_r}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right) = -c\,\frac{\vec{v}\,.\,\vec{R}_r}{cR_r - \vec{v}\,.\,\vec{R}_r} \quad \cdots\cdots DR(3)$$

$$= \frac{\vec{v}\,.\,\hat{R}_r}{1 - \frac{\vec{v}}{c}\,.\,\hat{R}_r} = -\vec{v}\,.\,\hat{R}_r\,\frac{\partial t_r}{\partial t} \quad \cdots\cdots DR(4)$$

Derivatives of the retarded variables

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\vec{v}\,.\,\vec{R}_r\right) &= \frac{\partial}{\partial x}\left[v_x\left(x-x'\right)+v_y\left(y-y'\right)+v_z\left(z-z'\right)\right] \\
&= \frac{\partial v_x}{\partial t_r}\frac{\partial t_r}{\partial x}\left(x-x'\right) + v_x\left(1-\frac{\partial x'}{\partial t_r}\frac{\partial t_r}{\partial x}\right) + \\
&\quad \frac{\partial v_y}{\partial t_r}\frac{\partial t_r}{\partial x}\left(y-y'\right) + v_y\left(-\frac{\partial y'}{\partial t_r}\frac{\partial t_r}{\partial y}\right) + \\
&\quad \frac{\partial v_z}{\partial t_r}\frac{\partial t_r}{\partial x}\left(z-z'\right) + v_z\left(-\frac{\partial z'}{\partial t_r}\frac{\partial t_r}{\partial x}\right) \\
&= \left(\vec{a}\,.\,\vec{R}_r\right)\frac{\partial t_r}{\partial x} + v_x - v^2\frac{\partial t_r}{\partial x}
\end{aligned}
$$

$$\nabla\left(\vec{v}\,.\,\vec{R}_r\right) = \left[\vec{a}\,.\,\vec{R}_r - v^2\right]\nabla t_r + \vec{v} \quad \cdots\cdots\cdots DR(5)$$

$$\frac{\partial}{\partial t}\left(\vec{v}\,.\,\vec{R}_r\right) = \left[\vec{a}\,.\,\vec{R}_r - v^2\right]\frac{\partial t_r}{\partial t} \quad \cdots\cdots\cdots DR(6)$$

Derivatives of the retarded variables

$$
\begin{aligned}
\left(\nabla \times \vec{v}\right)_i &= \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \\
&= \epsilon_{ijk} \frac{\partial v_k}{\partial t_r} \frac{\partial t_r}{\partial x_j} \\
&= \epsilon_{ijk} a_k \left(\nabla t_r\right)_j \\
\nabla \times \vec{v} &= \nabla t_r \times \vec{a}
\end{aligned}
$$

using $\cdots DR(1)$

$$
\boxed{\nabla \times \vec{v} = -\frac{1}{c} \frac{\vec{R}_r}{R - \frac{\vec{v}}{c} . \vec{R}} \times \vec{a} \; = \; \frac{1}{c} \frac{\vec{a} \times \vec{R}_r}{R - \frac{\vec{v}}{c} . \vec{R}} \qquad \cdots DR(7)}
$$

Now we can get what we want....

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{1-\frac{\vec{v}}{c}.\hat{R}_r}\right)\frac{1}{R_r} = \frac{qc}{4\pi\epsilon_0}\left(\frac{1}{c-\vec{v}.\hat{R}_r}\right)\frac{1}{R_r}$$

$$\nabla V = \frac{q}{4\pi\epsilon_0}\frac{-1}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2}\left[\nabla R_r - \frac{1}{c}\nabla(\vec{v}.\vec{R})\right]$$

Using $\cdots DR(1)$ and $\cdots DR(5)$

$$\boxed{\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2}\left[\frac{\vec{v}}{c} - \frac{\frac{\vec{a}.\vec{R}_r}{c^2} + \left(1-\frac{v^2}{c^2}\right)}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)}\vec{R}_r\right]}$$

Now we can get what we want....

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\vec{v}}{c^2}\left(\frac{1}{1-\frac{\vec{v}}{c}.\hat{R}_r}\right)\frac{1}{R_r} = \frac{q}{4\pi\epsilon_0}\frac{\vec{v}}{c}\left(\frac{1}{c-\vec{v}.\hat{R}_r}\right)\frac{1}{R_r}$$

$$\frac{\partial\vec{A}}{\partial t} = \frac{1}{c^2}\left(\frac{\partial\vec{v}}{\partial t_r}\right)\left(\frac{\partial t_r}{\partial t}\right)V + \frac{q}{4\pi\epsilon_0}\frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\left[\frac{1}{R_r-\frac{\vec{v}}{c}.\vec{R}_r}\right]$$

$$= \frac{\vec{a}}{c^2}V\left(\frac{\partial t_r}{\partial t}\right) - \frac{q}{4\pi\epsilon_0}\frac{\vec{v}}{c^2}\frac{1}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2}\left[\frac{\partial R_r}{\partial t}-\frac{1}{c}\frac{\partial}{\partial t}(\vec{v}.\vec{R}_r)\right]$$

The derivatives have been done in $\cdots DR(2)\cdots DR(3)\cdots DR(6)$

Finally $\vec{E}$ and $\vec{B}$

$$\frac{\partial A}{\partial t} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2}\left(\frac{\vec{a}}{c^2}R_r-\frac{\vec{v}}{c}\right)-\frac{\vec{v}}{c}\frac{R_r}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}.\vec{R}_r}{c^2}\right)\right]$$

$$\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2}\left[\frac{\vec{v}}{c} - \frac{\vec{R}_r}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}.\vec{R}_r}{c^2}\right)\right]$$

IMPORTANT : The terms associated with acceleration fall off as 1/R.
Notice that there are similar looking groups of terms

$$\vec{E}(\vec{r},t)=\frac{q}{4\pi\epsilon_0}\left[\frac{\left(1-\beta^2\right)\left(\hat{R}_r-\vec{\beta}\right)}{\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r^{\ 2}} + \frac{\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}}{c\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r}\right]$$

Coulomb

Radiation

Finally $\vec{E}$ and $\vec{B}$

$$\vec{B}(\vec{r},t) = \nabla\times\vec{A} = \nabla\times\frac{\vec{v}}{c^2}V(\vec{r},t)$$

$$= \frac{1}{c^2}\left[(\nabla\times\vec{v})V + (\nabla V)\times\vec{v}\right] \quad \cdots\cdots \text{use} \quad DR(7)$$

$$= \frac{V}{c^3}\,\frac{\vec{a}\times\vec{R}_r}{R-\frac{\vec{v}}{c}.\vec{R}_r} + (\nabla V)\times\frac{\vec{v}}{c^2}$$

Only this part can contribute

$$\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2}\left[\frac{\vec{v}}{c} - \frac{\vec{R}_r}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}.\vec{R}_r}{c^2}\right)\right]$$

$$\therefore\ \vec{B}(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{1}{c^2}\frac{\vec{a}\times\vec{R}_r}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^2} - \frac{1}{c}\frac{\vec{R}_r\times\vec{v}}{\left(R_r-\frac{\vec{v}}{c}.\vec{R}_r\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}.\vec{R}_r}{c^2}\right)\right]$$

Finally $\vec{E}$ and $\vec{B}$

We will now try to separate out the parts that depend on velocity and accelaration. For the E field, there were three parts – static Coulomb, velocity dependent only and then a part that depends on acceleration. For B there is no static part, since a charge at rest does not produce a magnetic field.

$$\vec{B} = \frac{q \quad \vec{R}_r \times}{4\pi\epsilon_0 c}\left[-\frac{\vec{v}/c}{\left(R_r - \frac{\vec{v}}{c}.\vec{R}_r\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}.\vec{R}_r}{c^2}\right) - \frac{\vec{a}/c^2}{\left(R_r - \frac{\vec{v}}{c}.\vec{R}_r\right)^2}\right]$$

$$= \frac{q \quad \vec{R}_r \times}{4\pi\epsilon_0 c}\left[-\frac{\vec{\beta}}{\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r^{\,3}}\left(1-\beta^2+\frac{\dot{\vec{\beta}}.\vec{R}_r}{c}\right) - \frac{1}{c}\frac{\dot{\vec{\beta}}}{\left(1-\vec{\beta}.\hat{R}_r\right)^2 R_r^{\,2}}\right]$$

Finally $\vec{E}$ and $\vec{B}$

$$\vec{B} = \frac{q}{4\pi\epsilon_0 c}\frac{1}{\left(1-\vec{\beta}\cdot\hat{R}_r\right)^3 R_r^{\,2}}\ \hat{R}_r\times\left[-\ \vec{\beta}\left(1-\beta^2+\frac{\dot{\vec{\beta}}\cdot\vec{R}_r}{c}\right)\ -\ \frac{\dot{\vec{\beta}}}{c}\left(R_r-\vec{\beta}\cdot\vec{R}_r\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0 c}\frac{1}{\left(1-\vec{\beta}\cdot\hat{R}_r\right)^3 R_r^{\,2}}\ \hat{R}_r\times\left[-\vec{\beta}\left(1-\beta^2\right)+\frac{1}{c}\left\{\dot{\vec{\beta}}(\vec{\beta}\cdot\vec{R}_r)-\vec{\beta}(\dot{\vec{\beta}}\cdot\vec{R}_r)-\dot{\vec{\beta}}R_r\right\}\right]$$

We can add terms proportional to R inside the brackets, since the cross product will give zero. Utilise this to complete the Coulomb and Radiation terms

$$= \frac{q}{4\pi\epsilon_0 c}\frac{1}{\left(1-\vec{\beta}\cdot\hat{R}_r\right)^3 R_r^{\,2}}\ \hat{R}_r\times\left[\ \left(\boxed{\hat{R}_r-\vec{\beta}}\right)\ \left(1-\beta^2\right)+\right.$$

$$\left.\frac{R_r}{c}\left\{\dot{\vec{\beta}}\left(\vec{\beta}\cdot\hat{R}_r\right)-\vec{\beta}\left(\dot{\vec{\beta}}\cdot\dot{R}_r\right)-\dot{\vec{\beta}}+\boxed{\hat{R}_r\left(\hat{R}_r\cdot\dot{\vec{\beta}}\right)}\right\}\right]$$

$$= \frac{q}{4\pi\epsilon_0 c}\frac{\hat{R}_r\times}{\left(1-\vec{\beta}\cdot\hat{R}_r\right)^3}\left[\frac{\left(\hat{R}_r-\vec{\beta}\right)\left(1-\beta^2\right)}{R_r^{\,2}}+\frac{1}{c}\frac{\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}}{R_r}\right] \qquad = \frac{\hat{R}_r}{c}\times\vec{E}$$

Finally $\vec{E}$ and $\vec{B}$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\left(1-\beta^2\right)\left(\hat{R}_r-\vec{\beta}\right)}{\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r^{\ 2}} + \frac{\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}}{c\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r}\right]$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{R}_r\times\vec{E}$$

We see the static + velocity + acceleration dependent parts clearly.

Notice that the radiation field exists only if the charge accelerates.

Allows immediate calculation of the Poynting vector at large R.

How much does an accelarating point charge radiate?

An accelerating charge must be losing energy continuously!

Radiation: the far field $\vec{E}\times\vec{B}$

The E & B fields are complicated when they are considered in totality. However to understand how much radiation is there we only need to consider the ~1/r terms and calculate the Poynting vector.

$$\begin{aligned} S &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left(\frac{\hat{R}_r}{c}\times\vec{E}\right) \\ &= \frac{1}{c\mu_0}\left[\hat{R}_r\left(\vec{E}\,.\,\vec{E}\right) - \vec{E}\left(\vec{E}\,.\,\hat{R}_r\right)\right] \end{aligned}$$

But the radiation field is directed along $\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}$

$$\Rightarrow \hat{R}_r\,.\,\vec{E}_{radiation} = 0 \qquad \& \qquad S_{radial} = \frac{1}{c\mu_0}E^2 = \frac{1}{Z}E^2$$

$Z \approx 377\,\Omega$ is the vacuum impedance (recall antenna...)

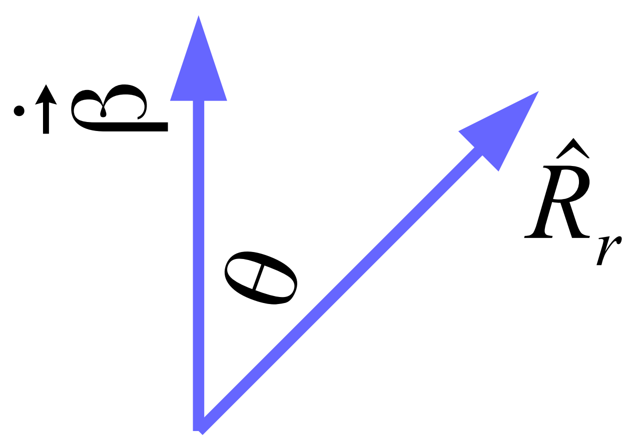
Do not confuse radiation field with radial component of the field !

Radiation: simplifying the far field $\vec{E}\times\vec{B}$

$$\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}\rightarrow\begin{cases} = & \hat{R}_r\times\hat{R}_r\times\dot{\vec{\beta}} & \text{if} \quad \vec{\beta}\parallel\dot{\vec{\beta}} \\ \approx & \hat{R}_r\times\hat{R}_r\times\dot{\vec{\beta}} & \text{if} \quad \vec{\beta}\ll 1 \end{cases}$$

Either velocity and acceleration are parallel
velocity is very small/charge at rest at (retarded) instant

In these cases $\quad |\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}|=\dot{\beta}\sin\theta=\dfrac{a}{c}\sin\theta$



Standard notation

$$\vec{\beta}=\frac{\vec{v}}{c} \quad \text{and} \quad \gamma=\frac{1}{\sqrt{1-\beta^2}}$$

Motion in a straight line

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\left(1-\beta^2\right)\left(\hat{R}_r-\vec{\beta}\right)}{\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r^{\,2}} + \frac{\hat{R}_r\times\left(\hat{R}_r-\vec{\beta}\right)\times\dot{\vec{\beta}}}{c\left(1-\vec{\beta}.\hat{R}_r\right)^3 R_r}\right]$$

$$|E_{radiation}|^2 = \left(\frac{q}{4\pi\epsilon_0 c}\right)^2\left[\frac{\dot{\beta}\sin\theta}{R_r\left(1-\beta\cos\theta\right)^3}\right]^2$$

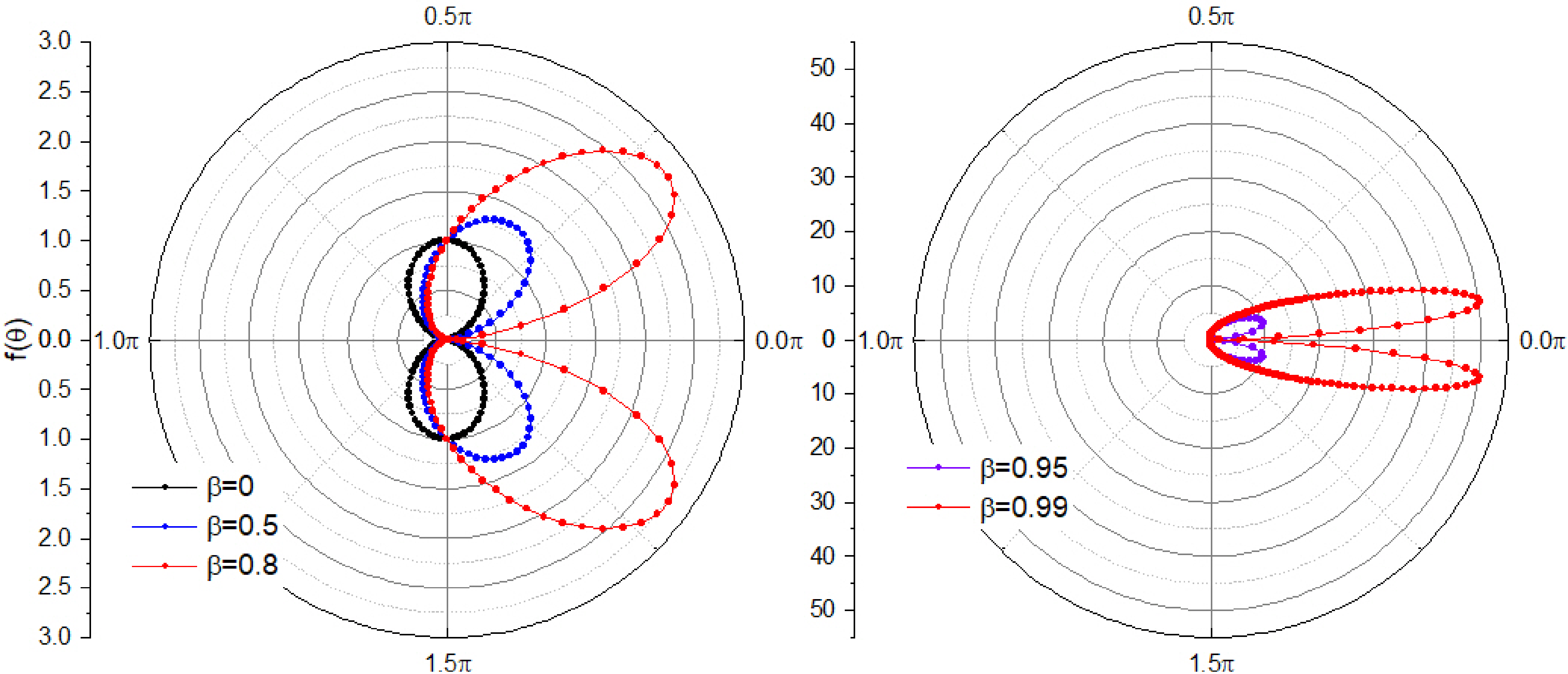
The radiated power

$$dP = S_r^{\,2} R^2 \sin\theta\, d\theta\, d\phi = S_r^{\,2} R^2 d\Omega$$

$$\frac{dP}{d\Omega} = \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\frac{q}{4\pi\epsilon_0 c}\right)^2\left[\frac{\dot{\beta}\sin\theta}{\left(1-\beta\cos\theta\right)^3}\right]^2 = \left(\frac{q^2}{16\pi^2\epsilon_0 c}\right)\left(\frac{a}{c}\right)^2 f(\theta)$$

The angular dependence of the radiation depends on the acceleration. The maximum will occur at angles determined by the acceleration.

The radiation pattern



The radiation concentrates around $\theta = 0$ as $\beta \rightarrow 1$

$$\cos\theta_{max} = \frac{\sqrt{1+24\beta} - 1}{4\beta}$$

$$P_{total} = \frac{1}{4\pi\epsilon_0}\frac{2q^2a^2}{3c^3}\left(1+\frac{\beta^2}{5}\right)\gamma^8$$

Energy loss rate due to radiation

The radiated power was measured by integrating over a large sphere at time t. This is NOT the loss rate of the accelerating particle.

δW passes through the spherical surface in time δt

But this was radiated by the charge between t_r and δt_r

$$\frac{\partial t_r}{\partial t} = \frac{cR_r}{cR_r - \vec{v}\,.\,\vec{R}_r} = \frac{1}{1 - \frac{\vec{v}}{c}\,.\,\hat{R}_r} \cdots\cdots\cdots\cdots DR(2)$$

$$P_r = \frac{\delta W}{\delta t_r} = \frac{\delta W}{\delta t}\frac{\delta t}{\delta t_r}$$

$$\frac{dP_r}{d\Omega} = \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\frac{q}{4\pi\epsilon_0 c}\right)^2\left[\frac{\dot{\beta}\sin\theta}{\left(1-\beta\cos\theta\right)^3}\right]^2\left(1-\beta\cos\theta\right)$$

Energy loss rate due to radiation

Completing the integral

$$P_{total} = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{2q^2a^2}{3c^3}}_{\text{Larmor}}\underbrace{\left(1+\frac{\beta^2}{5}\right)\gamma^6\left[1-\left(\frac{\vec{\beta}\times\dot{\vec{\beta}}}{\dot{\beta}}\right)^2\right]}_{\text{Lienard-Wiechart}}$$

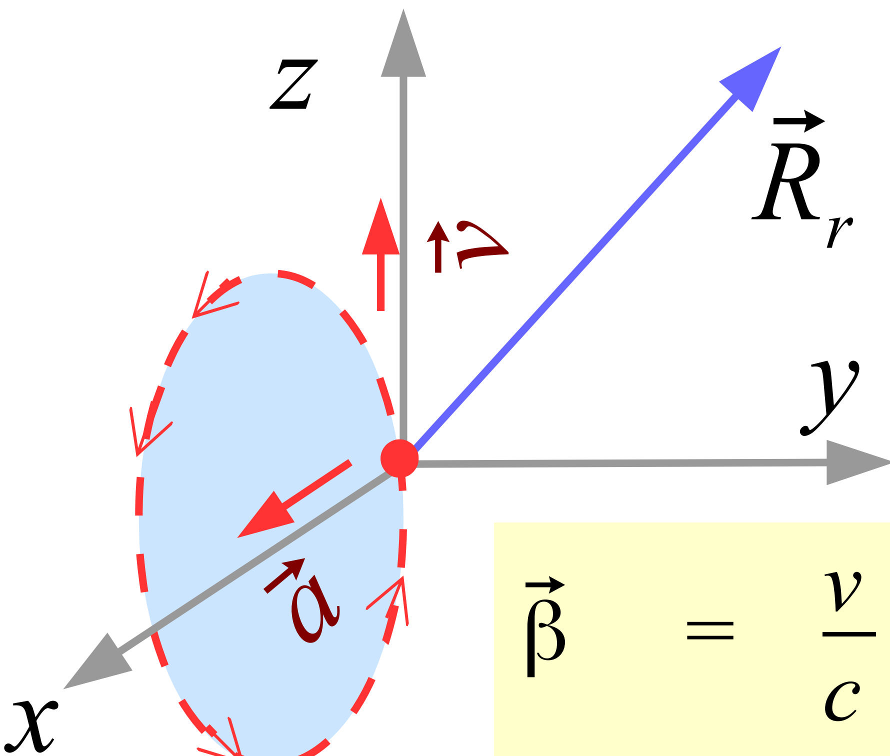
This radiation is the classical "Brehmsstralung" . The result does not say what the frequency distribution of the radiation is going to be. However the classical Brehmstralung has a flat frequency distribution upto a certain critical frequency.

Hitting a metal target with fast beam of electrons causes the electrons to decelerate rapidly. The energy is given off (partly) as X-ray with a continous spectrum. The characteristic X-ray lines (like Cu-Kα etc) arise from atomic transitions and are NOT brehmsstralung.

Circular motion : Synchrotron radiation

In circular motion (like an electron in a cyclotron) acceleration and velocity are perpendicular :

To ensure that the instantaneous motion is along z, we can take the orbit to be in the x-z plane, so that acceleration is instantaneously along x. (simplifies the algebra a bit!)



Evaluate

$$\hat{R}_r \times \left(\hat{R}_r - \vec{\beta}\right) \times \dot{\vec{\beta}}$$

$$\vec{\beta} = \frac{v}{c}\hat{k} \qquad \dot{\vec{\beta}} = \frac{a}{c}\hat{i}$$

$$\hat{R}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

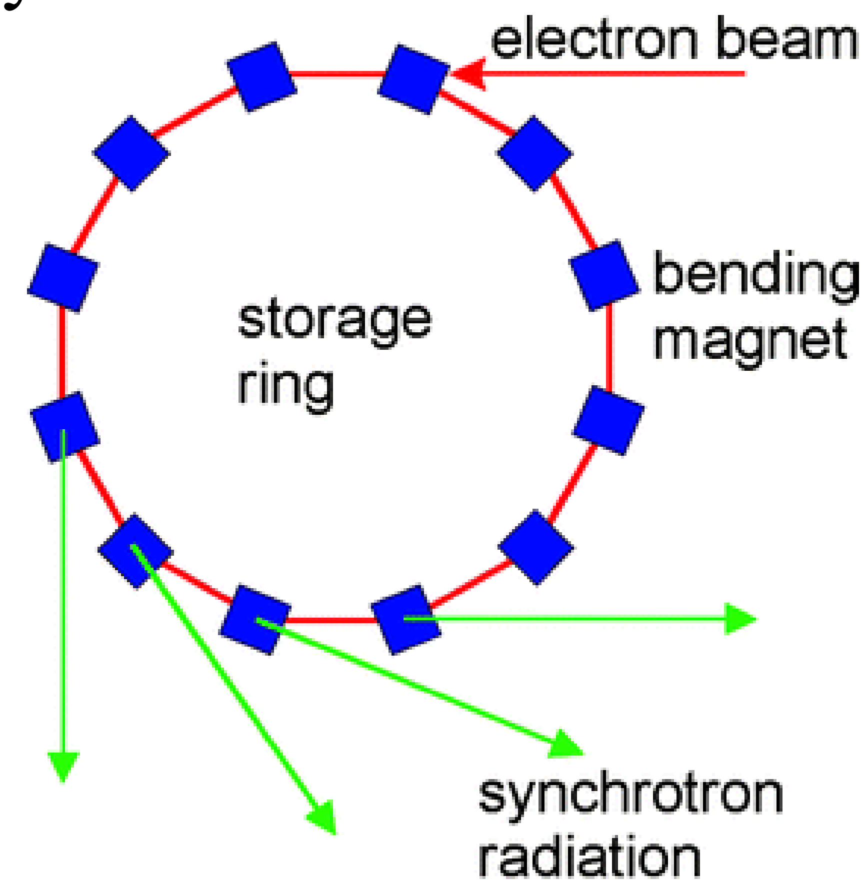
Circular motion : Synchrotron radiation

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{q^2 a^2}{4\pi c^3}\frac{(1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}$$

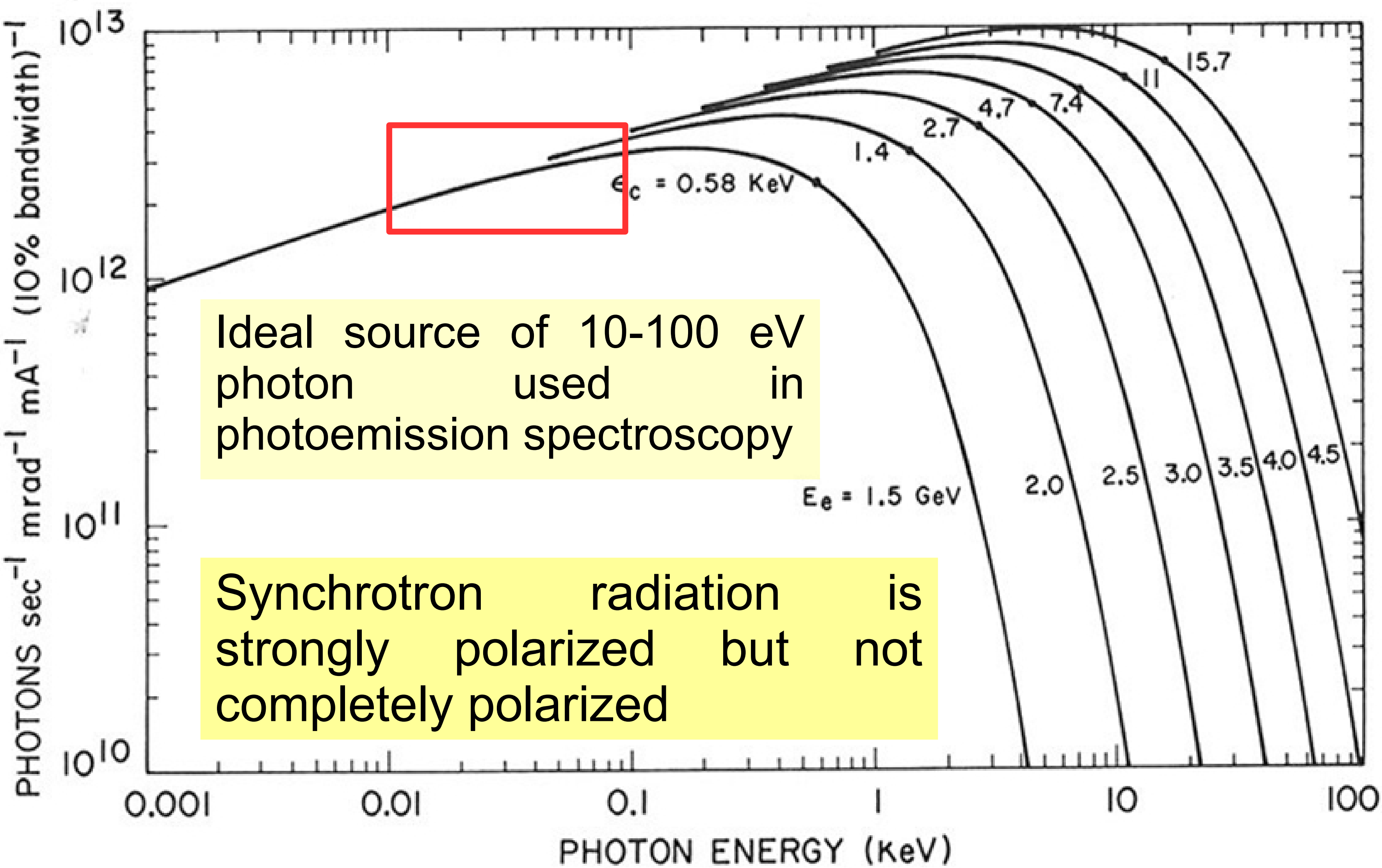
Radiation peaks in a direction normal to the acceleration
That means it is along the velocity

$$P_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2q^2 a^2}{3c^3}\gamma^4$$

Continuous freq distribution
cutoff at cyclotron frequency
determined by $\omega_c = \frac{eB_\perp}{m}$



Spectrum of synchrotron radiation



J.D. Jackson, 'Classical Electrodynamics", CC BY 2.5,
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Fourier components of the potential

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad \& \quad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}$$

In this the speed c has no special significance
The mathematical form of the solutions will be the same ....
With $\epsilon_0 \to \epsilon \quad \mu \approx \mu_0 \quad c \to c/n \qquad n$ is the refractive index

$$\tilde{V}(\vec{r},\omega) = \int d\tau' \left(\frac{\tilde{\rho}(\vec{r}\,',\omega)}{\epsilon}\right)\left[\frac{1}{4\pi|\vec{r}-\vec{r}\,'|} e^{\pm i(n\omega/c)|\vec{r}-\vec{r}\,'|}\right]$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon}\int d\tau' \frac{1}{|\vec{r}-\vec{r}\,'|}\rho\left(\vec{r}\,', t \pm \frac{|\vec{r}-\vec{r}\,'|}{c/n}\right)$$

$$\frac{1}{1-\frac{\vec{v}}{c}.\hat{R}} \to \frac{1}{1-n\frac{\vec{v}}{c}.\hat{R}}$$

The Lienard-Wiechart factor can now diverge because the speed of a particle CAN be greater than c/n in a medium.

Fourier components of the potential

So far we have written explicit times dependent expressions for E and B. But this is often not useful or necessary. Since we are dealing with waves and radiation, the problems are often better handled in terms of the Fourier component.

We often ask questions like how much power is radiated within a spectral band f to f+df for example.

If we retain the first expression (in terms of angular frequency only, we almost do not need to talk about "retarded time" because the integrals will be over space. Solutions for V and A will be similar, becuase the basic equations are similar.

The divergence of the Lienard-Wiechart will happen only at a specific angle! The consequence is that a charge moving with a constant velocity in a medium with a speed greater than c/n can radiate at that specific angle. This radiation is called the Cerenkov radiation.

Important equations in Fourier language

Continuity

$$\nabla . \vec{j} + \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \nabla' . \vec{j}(\omega) - i\,\omega\,\rho(\omega) = 0$$

Lorenz gauge

$$\nabla . \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \quad \Rightarrow \quad \nabla . \vec{A}(\omega) - i\,\frac{\omega}{c^2} V(\omega) = 0$$

Notice the differentiation in the first equation. The divergence of the current muct be calculated w.r.t. The source co-ordinates (primed). All the other derivatives are w.r.t the observation point.

The continuity equation ensures that the the charge density and current cannot vary in an arbitrary way. This must always be ensured.

The standard derivates to get the E and B fields from V and A can now be carried out....

Important equations in Fourier language

$$\vec{E}(\vec{r},\omega) = \frac{1}{4\pi\epsilon_0}\left[\int \rho(\vec{r}\,',\omega)\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}e^{ik|\vec{r}-\vec{r}\,'|}d\tau' - \right.$$

$$\boxed{k=\frac{\omega}{c}} \qquad \left. ik\int\left(\rho(\vec{r}\,',\omega)\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} - \frac{\vec{j}(\vec{r}\,',\omega)}{c}\right)\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}d\tau'\right]$$

$$\vec{B}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\left[\int \frac{\vec{j}(\vec{r}\,',\omega)\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}e^{ik|\vec{r}-\vec{r}\,'|}d\tau' - \right.$$

$$\left. ik\int \frac{\vec{j}(\vec{r}\,',\omega)\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}e^{ik|\vec{r}-\vec{r}\,'|}d\tau'\right]$$

Can you recover the simplest "static" solutions ?
Which terms would give radiation ? Can you calculate E x B ?
What could be the advantage of writing it this way?

Important equations in Fourier language

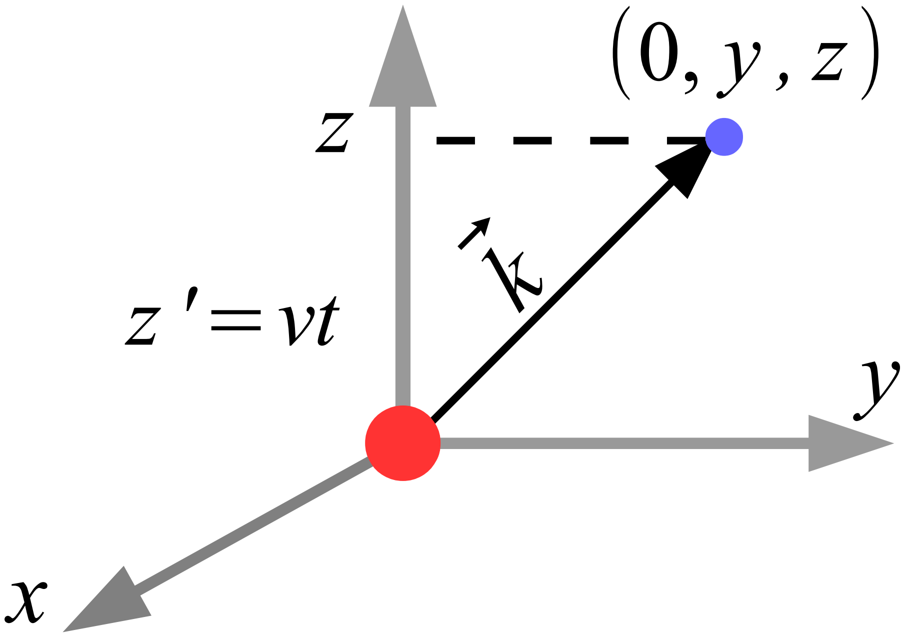
We now take the usual route of using the 1/r part of E and B to calculate the Poynting vector and the radiated power. The calculations do not have retarded time explicitly but are still quite long......The final result is very useful ! The additional information we extract from it is the spectral dependence of the radiation.

Define $\vec{k} = \frac{\omega}{c} \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|}$ & $d\Omega$ solid angle along $\vec{k}$

$$\frac{dU(\omega)}{d\Omega} d\omega = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \left| \int \left(\vec{j}(\omega) \times \vec{k} \right) e^{i\vec{k}.\vec{r}\,'} d\tau\,' \right|^2 d\omega$$

The integral runs over source co-ordinates only

The $\check{C}erenkov$ contribution



The charge moves with uniform velocity along z axis. We saw before that it does NOT radiate. There is no acceleration.

Things change if the region is partly filled with a material of refractive index (n) between -L < z < L

$$\begin{aligned}
\vec{j}(\vec{r}', t) &= q\vec{v}\,\delta(x')\delta(y')\delta(z'-vt) \\
\vec{j}(\omega) &= \int_{-\infty}^{\infty} q\vec{v}\,\delta(x')\delta(y')\delta(z'-vt)e^{i\omega t}dt \\
&= q\,v\,\hat{z}\,\delta(x')\delta(y')e^{i\omega z'/v}\cdot\frac{1}{v} \\
&= q\,\delta(x')\delta(y')e^{i\omega z'/v}\,\hat{z}
\end{aligned}$$

Evaluating the $\check{C}erenkov$ contribution

$$I(\omega) = \int \left(\vec{j}(\omega)\times\vec{k}\right)e^{-i\vec{k}.\vec{r}'}d\tau'$$

$$= \hat{\epsilon}_\phi \int \sin\theta \;\; \delta(x')\delta(y')e^{i\omega z'/v}\times$$

$$e^{-i\left(k\sin\theta\cos\phi\, x' + k\sin\theta\sin\phi\, y' + k\cos\theta\, z'\right)}dx'dy'dz'$$

$$= \hat{\epsilon}_\phi \frac{qnv}{c}\sin\theta \int_{z'=-L}^{z'=L} e^{i(\omega z'/v)(1-nv/c\cos\theta)} d\left(\frac{\omega z'}{v}\right)$$

$$\left|I(\omega)\right|^2 = \frac{q^2n^2v^2}{c^2}\sin^2\theta\left|\int_{z'=-L}^{z'=L} e^{i\xi\left(1-\frac{nv}{c}\cos\theta\right)}d\xi\right|^2 \cdots \xi = \frac{\omega z'}{v}$$

Compare with $\int_{-\infty}^{\infty} e^{i\xi p}d\xi = 2\pi\delta(p) \cdots\cdots p = 1-\frac{nv}{c}\cos\theta$

This can contribute only if $\frac{nv}{c} > 1 \Rightarrow v > \frac{c}{n}$

Evaluating the $\check{C}erenkov$ contribution

$$\frac{dU(\omega)}{d\Omega}d\omega = \frac{1}{4\pi n}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\int\left(\vec{j}(\omega)\times\vec{k}\right)e^{i\vec{k}.\vec{r}'}d\tau'\right|^2 d\omega$$

The extra factor of n accounts for the index of the medium.

$$U(\omega) = \frac{1}{4\pi n}\sqrt{\frac{\mu_0}{\epsilon_0}}\int|I(\omega)|^2 d\Omega$$

$$= \frac{1}{4\pi n}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{q^2 n^2 v^2}{c^2}\left(\frac{2\omega L}{v}\right)^2 2\pi\int\sin^2\theta\left[\frac{\sin\left(\frac{\omega Lp}{v}\right)}{\frac{\omega Lp}{v}}\right]^2 d(\cos\theta)$$

delta-fn like term $\Rightarrow \sin^2\theta \approx 1-\frac{c^2}{n^2v^2}$ also $d(\cos\theta)=\frac{c}{nv}dp=\frac{c}{n\omega L}d\left(\frac{\omega Lp}{v}\right)$

Evaluating the $\check{C}erenkov$ contribution

$$U(\omega) = \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{q^2 n v^2}{c^2}\left(\frac{2\omega L}{v}\right)^2\left(1-\frac{c^2}{n^2v^2}\right)\frac{c}{n\omega L}\times\cdots$$

$$\cdots\int\left[\frac{\sin\left(\frac{\omega Lp}{v}\right)}{\frac{\omega Lp}{v}}\right]^2 d\left(\frac{\omega Lp}{v}\right)$$

Let $L\to\infty$, since contrib comes from a small region only

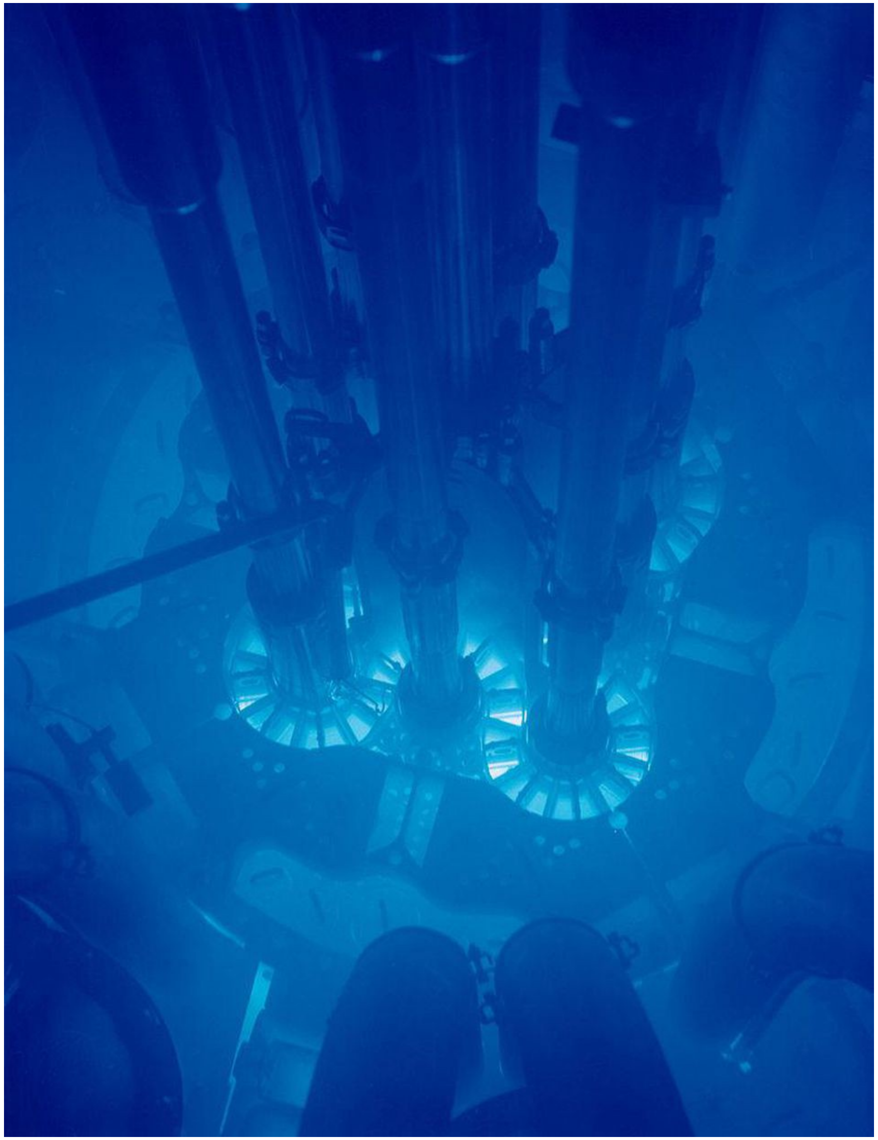
So We can use $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,dx=\pi$

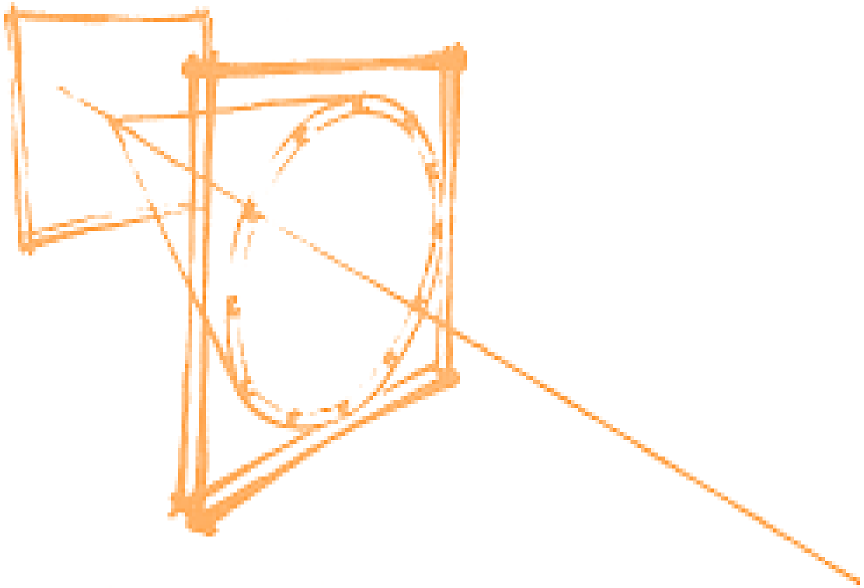
$$\frac{U(\omega)}{L}d\omega = \frac{2\pi q^2}{\epsilon_0 c^2}\left(1-\frac{c^2}{n^2v^2}\right)\omega\, d\omega \quad \cdots\cdots\cdots \text{radiated per unit length}$$

Rewrite the expression in terms of quanta/unit length

$$\frac{\Delta N}{\Delta L}d\omega = \alpha\left(1-\frac{c^2}{n^2v^2}\right)\frac{d\omega}{c}\cdots\cdots\alpha=\frac{1}{137}$$

Evaluating the $\check{C}erenkov$ contribution





$$\cos\theta = \frac{c}{nv}$$

Pic: Argonne National Lab (high speed electrons in water surrounding a nuclear reactor core)

Radiation reaction

An accelerated charge radiates → it must be losing energy → this should affect its motion by slowing it down → We should be able to write its equation of motion.

Simple expectation...but it has fundamental difficulties!

We are ignoring the energy that might go back and forth between the "particle" and the nearfield/velocity field.

Recall the energy lost by radiation

$$P_{total} = \frac{1}{4\pi\epsilon_0}\frac{2q^2a^2}{3c^3}\left(1+\frac{\beta^2}{5}\right)\gamma^6\left[1-\left(\frac{\vec{\beta}\times\dot{\vec{\beta}}}{\dot{\beta}}\right)^2\right]$$

$$= \frac{q^2}{6\pi\epsilon_0 c^3}a^2 \qquad \cdots\cdots(\text{for } v \ll c)$$

$$\Rightarrow \vec{F}_{react}.\vec{v} + \frac{q^2}{6\pi\epsilon_0 c^3}\dot{\vec{v}}^2 = 0$$

How much is the *loss*?

An electron accelerates for time T from rest with acceleration a

$$K.E. = \frac{m(aT)^2}{2} \quad \& \quad E_{rad} = P_{rad}T = \frac{e^2a^2}{6\pi\epsilon_0 c^3}T$$

$$\frac{E_{rad}}{K.E.} = \frac{e^2a^2T}{6\pi\epsilon_0 c^3}\cdot\frac{2}{m\,a^2T^2} = \frac{1}{6\pi}\left(\frac{e^2}{\epsilon_0 mc^3}\right)\frac{1}{T} \approx \frac{10^{-24}}{T}$$

$\Rightarrow$ for $T > 10^{-24}\,sec$ $\rightarrow$ Loss is a small perturbation

For circular (cyclotron) motion we calculate loss/period (T)

$$\frac{E_{rad}}{K.E.} = \frac{4\pi}{3}\left(\frac{e^2}{\epsilon_0 mc^3}\right)\frac{1}{T} \approx \frac{10^{-23}}{T}$$

The timescale is similar to what light needs to cross a typical nucleus! Nucleus has size ~ 10^{-15} m, divide by c ~ 10^{8} m/s.

How much is the *loss*?

Integrate $\int \dot{\vec{v}} \, . \, \dot{\vec{v}} \, dt$ by parts from t_1 to t_2

$$\int_{t_1}^{t_2} \boxed{\left(\vec{F}_{react} - \frac{q^2 \ddot{\vec{v}}}{6\pi\epsilon_0 c^3} \right)} . \vec{v} \, dt + \left[\frac{q^2}{6\pi\epsilon_0 c^3} \vec{v} \, . \, \dot{\vec{v}} \right]_{t_1}^{t_2} = 0$$

For an arbitrary path there is no correlation between velocity and acceleration. But only if the motion is periodic and we integrate over one period, then the second term can be exactly zero.

We CLAIM that the integrand in the first term is zero on average....
This is the non-relativistic Abraham-Lorenz formula.

However, since it is a dot product nothing can be said for components of F perpendicular to v.

How much is the *loss*?

What is the microscopic origin of this reaction/retardation (shown by Lorenz) lies in the retarded fields created by one part of the object on the other parts.

For any finite object moving with a rigid acceleration these "internal" forces DO NOT cancel.

The exact coefficient of the the da/dt term depends on the geometry.

Also an additional mass term comes from the fact that the internal electric fields carry energy. The object that accelerates is thus (rest mass + some energy contained in the fields). The generic form of the equation of motion is as follows.

$$a = \frac{f_{ext}(t)}{m_0 + m_{field}} + \tau \dot{a} \qquad \begin{cases} \tau \approx \dfrac{q^2}{6\pi\epsilon_0 m c^3} \\ \quad \sim 10^{-24}\,\text{sec} \quad \text{for an electron} \end{cases}$$

How much is the *loss*?

The generic solution of the equation has an unexpected feature.
To write down the formal solution for an arbitrary f(t) we can use the Green's function method.
It is also possible to write down the integrating factor directly....

$$a-\tau\dot{a} = \frac{F_{ext}(t)}{m} \equiv f(t) \quad \cdots \quad \text{where } m=m_0+\text{some bit}$$

$$G-\tau\dot{G} = \delta(t-t')$$

$$G(t-t') = \begin{cases} A_1 e^{t/\tau} & \cdots \quad t<t' \\ A_2 e^{t/\tau} & \cdots \quad t>t' \end{cases} \quad \cdots \quad G(t'+\epsilon)-G(t'-\epsilon)=-\frac{1}{\tau}$$

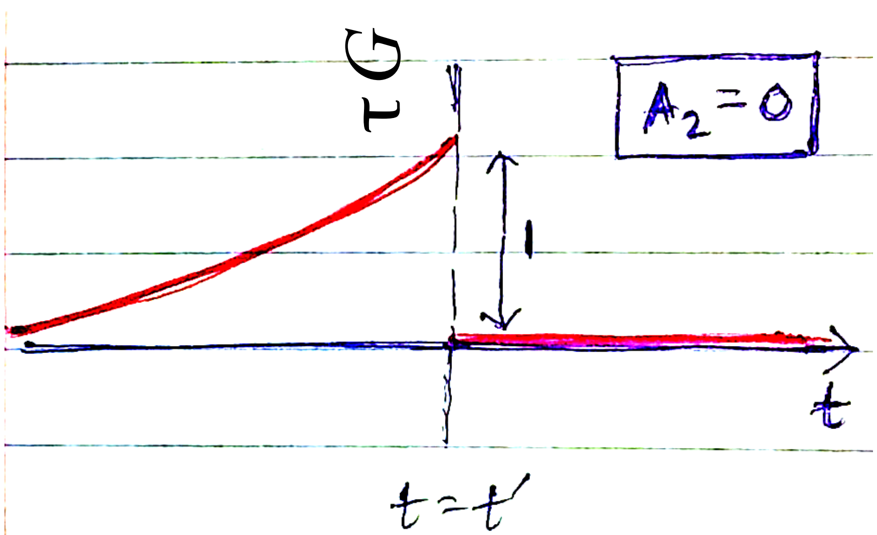
$$\Rightarrow \quad A_1-A_2=\frac{e^{-t'/\tau}}{\tau}$$

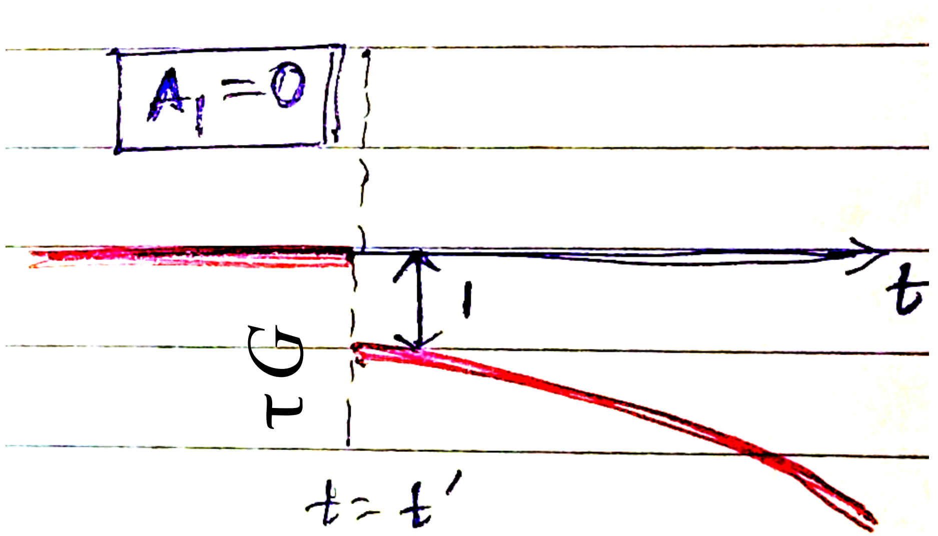
Solving the acceleration equation

$$A_1 - A_2 = \frac{e^{-t'/\tau}}{\tau} \Rightarrow \text{either } A_1 = 0 \quad \text{OR} \quad A_2 = 0$$

$$A_2 = 0 \Rightarrow G(t-t') = \begin{cases} \dfrac{e^{(t-t')/\tau}}{\tau} & \cdots\cdots \quad t < t' \\ 0 & \cdots\cdots \quad t > t' \end{cases}$$

$$A_1 = 0 \Rightarrow G(t-t') = \begin{cases} 0 & \cdots\cdots \quad t < t' \\ -\dfrac{e^{(t-t')/\tau}}{\tau} & \cdots\cdots \quad t > t' \end{cases}$$





Non-causal implication of the solution

The solution that blows up as $t \to \infty$ is not acceptable.

$$A_2 = 0 \;\Rightarrow\; G(t-t') = \begin{cases} \dfrac{e^{(t-t')/\tau}}{\tau} & \cdots\cdots \quad t' > t \\ 0 & \cdots\cdots \quad t' < t \end{cases}$$

$$a(t) = \int_{-\infty}^{\infty} f(t')\,G(t-t')\,dt' + \cancel{\text{homogeneous soln}}$$

$$= \int_{-\infty}^{t} f(t') \times 0 \; dt' + \frac{1}{\tau}\int_{t}^{\infty} f(t')\,e^{(t-t')/\tau}\,dt'$$

$\Rightarrow a(t)$ depends on $f(t)$ at future times !

upto a time of order $\tau \sim 10^{-24}$ sec

This peculiar discrepancy is NOT a calculation error

But has no obvious observational consequence